Encyclopaedia

Metropolitana.

First Division.

Pure Sciences.

30

"*The greatest and most important Literary Enterprise of the day.*"—CRITIC.

CABINET EDITION OF THE

ENCYCLOPÆDIA METROPOLITANA,

Thoroughly Revised and abundantly Illustrated,

Publishing in Volumes, each containing a Complete Treatise, Crown Octavo, Cloth.

Volumes now Ready.

Vol. I., price 2s., cloth lettered,
INTRODUCTORY DISSERTATION ON THE SCIENCE OF METHOD. By SAMUEL TAYLOR COLERIDGE. With a Synopsis. Sixth Edition.

Vol. II., price 5s., cloth lettered,
UNIVERSAL GRAMMAR, or the Pure Science of Language. By Sir JOHN STODDART, Knt., LL.D. Third Edition, Revised.
"The soundest and most philosophical treatise of the kind in the English language."—*North British Review.*

Vol. III., price 3s., cloth lettered,
LOGIC. By the Most Rev. RICHARD WHATELY, D.D., Archbishop of Dublin. The Original Edition, with Synopsis and Index by the Editor.

Vol. IV., price 3s. 6d., cloth lettered,
RHETORIC. By the Most Rev. RICHARD WHATELY, D.D., Archbishop of Dublin. The Original Edition, with Synopsis and Index by the Editor.

Vol. V., price 6s., cloth lettered,
HISTORY OF THE CHRISTIAN CHURCH in the First Century. By the Right Rev. SAMUEL HINDS, D.D., Bishop of Norwich, and the Rev. J. H. NEWMAN, D.D. Third Edition, Revised.
"Erudite research is here combined with independent thought."—*Scottish Guardian.*

Vol. VI., price 4s., cloth lettered,
POLITICAL ECONOMY. By NASSAU WILLIAM SENIOR, Esq., late Professor of Political Economy in the University of Oxford. Third Edition.

Vol. VII., price 2s. 6d., cloth lettered,
HISTORY OF THE JEWS, from the time of Alexander the Great to the Destruction of Jerusalem by Titus, A.M. 3595, B.C. 409 to A.D. 70. By the Venerable Archdeacon HALE, A.M., Master of the Charter-house. Fourth Edition.

Vol. VIII., price 6s., cloth lettered,
SACRED HISTORY AND BIOGRAPHY, from the Antediluvian Period to the Time of the Prophet Malachi, A.M. 1 to A.M. 3607, B.C. 397. Edited by the Rev. F. A. COX, D.D., LL.D.
"Pleasing and good, and worthy of general attention."—*United Presbyterian Magazine.*

Vol. IX., price 7s. 6d., cloth lettered,
HISTORY OF GREEK LITERATURE. By the Hon. Sir T. N. TALFOURD, D.C.L., the Right Rev. C. J. BLOMFIELD, D.D., Bishop of London; R. WHITCOMBE, Esq., M.A., Trinity College, Cambridge; E. POCOCKE, Esq.; the Rev. J. B. OTTLEY, M.A.; and the Rev. H. THOMPSON, M.A., St. John's College, Cambridge.
"Full of information, expressed in the choicest language."—*Athenæum.*

Vol. X., price 5s., cloth lettered,
MORAL AND METAPHYSICAL PHILOSOPHY.—SYSTEMS OF PHILOSOPHY ANTERIOR TO THE TIME OF CHRIST. By the Rev. F. D. MAURICE, M.A. Third Edition. Re-written.
"Complete and satisfactory."—*Literary Gazette.*

Vol. XI., price 5s., cloth lettered,
INTRODUCTION TO UNIVERSAL HISTORY—ON THE USES OF HISTORY AS A STUDY, and ON THE SEPARATION OF THE EARLY FACTS OF HISTORY FROM FABLE. By Sir JOHN STODDART, LL.D. R
"We cannot speak too highly in its p

Cabinet Edition of the Encyclopædia Metropolitana—continued.

Vol. XII., price 8s. 6d., cloth lettered,

ROMAN ANTIQUITIES. By WILLIAM RAMSAY, M.A., Professor of Humanity in the University of Glasgow. With Map, and One Hundred and Thirty Illustrations. Third Edition.

"Comprises all the results of modern improved scholarship within a moderate compass."—*Athenæum.*

Vol. XIII., price , cloth lettered,

BOTANY. By JOHN HUTTON BALFOUR, M.D., F.R.S.E., Professor of Botany in the University of Edinburgh. Numerous Engravings. Third Edition. *In Preparation.*

"Perhaps the most masterly digest of the science which has yet appeared."—*Witness.*

Vol. XIV., price 3s. 6d., cloth lettered,

ELECTRO-METALLURGY: Containing an Account of the most Improved Methods of Depositing Copper, Silver, Gold, and other Metals. Second Edition, with Numerous Illustrations. By JAMES NAPIER, F.C.S.

"With this book at hand, we don't think it possible to fail in obtaining satisfactory results."—*Art Journal.*

Vol. XV., price 9s., cloth lettered,

HISTORY OF GREECE from the Earliest Times to the Close of the Peloponnesian War. By E. POCOCKE, Esq.; Sir T. N. TALFOURD, D.C.L.; Rev. J. B. OTTLEY, A.M.: and the late J. T. RUTT, Esq. With upwards of One Hundred Engravings, from the most authentic sources.

"Altogether a gem."—*Glasgow Citizen.*

Vol. XVI., price 6s., cloth lettered,

PHOTOGRAPHY. By ROBERT HUNT, F.R.S., Professor of Physical Science in the Metropolitan School of Science, London. Fourth Edition, Numerous Engravings.

"Must prove of infinite service to those engaged in the pursuit of this entertaining science."—*Art Journal.*

Vol. XVII., price 3s., cloth lettered,

VETERINARY ART: A Practical Treatise on the Diseases of the Horse. By W. C. SPOONER, Esq., with 50 Engravings.

Vol. XVIII., price 8s., cloth lettered,

EARLY ORIENTAL HISTORY, comprising the History of the Egyptians, Assyrians, Phœnicians, Persians, &c. By the Rev. Professor EADIE, D.D., LL.D. With Numerous Illustrations.

"In all respects a thoroughly good book."—*Dr. Kitto.*

Vol. XIX., price 8s. 6d., cloth lettered,

HISTORY OF THE ROMAN REPUBLIC. By the Rev. THOS. ARNOLD, D.D., late Head Master, Rugby School; Sir THOS. N. TALFOURD, D.C.L.; the Rev. Professor JEREMIE, and others. With Numerous Illustrations.

Vol. XX., price 7s. 6d., cloth lettered,

BIBLICAL ANTIQUITIES AND GEOGRAPHY, including the Language, Geography, and Early History of Palestine. Edited by the Rev. F. A. COX, D.D., LL.D. With Maps and Numerous Illustrations.

"The most readable book on the subject in our language."—*Baptist Magazine.*

Vol. XXI., price 12s. 6d., cloth lettered,

METALLURGY. A Practical Treatise on the Chemistry of the Metals. By JOHN ARTHUR PHILLIPS, Esq., F.C.S. Second Edition, revised. Illustrated by nearly 200 Engravings.

"A *sine qua non* to every metallurgist."—*Mining Journal.*

Vol. XXII., price 4s., cloth lettered,

HISTORY OF THE CHRISTIAN CHURCH in the Second and Third Centuries. By JAMES AMIRAUX JEREMIE, D.D., Regius Professor of Divinity, Cambridge.

"It is an admirable text-book."—*Evangelical Magazine.*

Vol. XXIII., price 8s., cloth lettered,

HISTORY OF GREECE AND MACEDONIA, from the Age of Xenophon to the absorption of Greece in the Roman Empire, including the History of Alexander the Great and his Successors. By Dr. LYALL, Dean of Canterbury, Rev. G. C. RENOUARD, B.D., &c. &c. With Numerous Illustrations.

"The dissertations are full, luminous, and charmingly written."—*Eclectic Review.*

Vol. XXIV., price 10s. 6d., cloth lettered,

HISTORY OF ROMAN LITERATURE. By the Rev. THOS. ARNOLD, D.D., late Head Master, Rugby School; Rev. HENRY THOMPSON, M.A.; Rev. Dr. NEWMAN; Rev. J. B. OTTLEY, M.A.; and Rev. J. M. NEALE, Warden of Sackville College.

2/6

399 e. $\frac{265}{30}$

ENCYCLOPÆDIA METROPOLITANA:

OR,

A SYSTEM OF UNIVERSAL KNOWLEDGE:

ON A METHODICAL PLAN,

PROJECTED BY SAMUEL TAYLOR COLERIDGE.

CABINET EDITION.

TRIGONOMETRY.

A TREATISE

ON

TRIGONOMETRY.

BY

GEORGE BIDDELL AIRY, M.A., F.R.S.,
ASTRONOMER ROYAL.

NEW EDITION, REVISED, BY

HUGH BLACKBURN, M.A.,
PROFESSOR OF MATHEMATICS IN THE UNIVERSITY OF GLASGOW.

LONDON AND GLASGOW:
RICHARD GRIFFIN AND COMPANY,
PUBLISHERS TO THE UNIVERSITY OF GLASGOW.
1855.

ADVERTISEMENT.

THE TREATISE on TRIGONOMETRY by Mr. AIRY was written expressly for the ENCYCLOPÆDIA METROPOLITANA. The author's engagements, as Astronomer Royal, leaving him little leisure, he was unable to make those amendments in the work that were required by the plan of the Revised Edition of the ENCYCLOPÆDIA. The Treatise was therefore placed in the competent hands of Professor BLACKBURN; who, in superintending its passing through the press, has added questions, and made such other improvements as, it is hoped, will render it more suitable both for a class book and for private study.

CONTENTS.

TRIGONOMETRY.

TRIGONOMETRY (Τριγωνομετρία, from τριγωνὸς, a triangle, and μετρέω, I measure), the science of triangles, the branch of mathematics which treats of the application of arithmetic to geometry. The term was originally restricted to signify the science which gives the relation of the parts of triangles described on a plane or spherical surface; but it is now understood to comprehend all theorems respecting the properties of angles and circular arcs, and the lines belonging to them. This latter department is frequently called the arithmetic of sines.

In the application of mathematics to physics, no branch is more important than trigonometry. It is the connecting link by which we are enabled to combine, in their fullest extent, the practical exactness of arithmetical calculations with the hypothetical accuracy of geometrical constructions. Without it, the former could never have been applied to physics, and the limit of the errors of the latter would have depended on the skill of the practical geometer. By the substitution of numerical calculations for graphical constructions, we are enabled to obtain results to any desired degree of accuracy. With trigonometry, in fact, astronomy first received such a degree of exactness as justly to merit the name of science; and every improvement that has been made in trigonometry to the present time, has been attended with corresponding improvements in all parts of physical science.

The following will be the arrangement of the present treatise:— The first section will contain the definitions of the terms most frequently in use; in the second will be given the principal theorems relating to trigonometrical lines; the third will explain the use of subsidiary angles; the fourth will contain all the most important propositions of plane trigonometry; the fifth, those of spherical geometry; and the sixth, those of spherical trigonometry. In the seventh will be given formulæ for small corresponding variations of the parts of triangles; and the eighth will contain some theorems which require for their investigation a more refined analysis. The ninth will treat of some expressions peculiar to geodetic operations; and the tenth will explain the construction of trigonometrical tables.

SECTION I.

DEFINITIONS.

(1.) LET A B (fig. 1) be a circular arc, of which C is the centre, and let C A, C B be joined. The arc A B is proportional to the angle A C B, and either of these can therefore be used as the measure of the other, provided the arc A B is less than half the circumference, or the angle A C B less than two right angles. Since this holds with regard to all the angles of triangles, we shall, in treating of them, use indifferently the terms *arc* and *angle* to express the inclination of two lines.

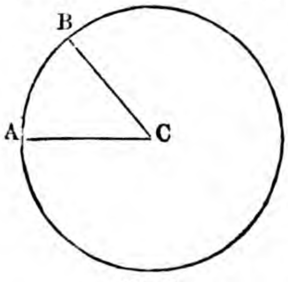

Fig. 1.

(2.) But in the higher parts of the science it is by no means a matter of indifference which term we employ. It is evident that an arc can be conceived to exceed, not only half a circumference, but even a whole circumference, or any number of circumferences; while an angle cannot be greater than two right angles. Much obscurity has frequently arisen from neglecting to observe, that when we speak of an angle greater than two right angles, we mean merely an arc greater than half a circumference; and that, when we consider trigonometrical lines as functions of such an angle, we intend nothing more than that they are functions of the corresponding arc of a circle. The reader, therefore, will be careful to recollect, that all trigonometrical lines are considered to be functions of the circular arc to which they correspond, the radius being given; and that there is no limit whatever to the extension of this arc.

(3.) The circumference of the circle has usually been divided into 360 equal parts, called *degrees;* each of these subdivided into 60, called *minutes;* each of these into 60, called *seconds;* the seconds are sometimes divided each into 60 *thirds*, the thirds into 60 *fourths*, &c., but they are more usually divided decimally. But in most of the French treatises lately published the circumference is divided into 400 equal parts, or *grades*, each grade into 100 minutes, and each minute into 100 seconds. Degrees, minutes, and seconds are commonly marked $^\circ$, $'$, $''$; grades and their subdivisions sometimes thus, ^{g}, $^{`}$, $^{``}$. Thus, $38^\circ\ 17'\ 22''$ is read thirty-eight degrees, seventeen minutes, twenty-two seconds; $44^g\,76'\,27''$, or $44^g,7627$, is forty-four grades, seventy-six grade-minutes, twenty-seven grade-seconds.

(4.) In most of the following investigations we shall consider the radius of the circle as the unit of linear measure. The semi-circumference is then $= 3,141592653590$; its logarithm $= 0,4971498726$; one degree $= 0,017453292520$; one minute $= 0,000290888209$; one second $= 0,000004848137$; their logarithms increased by 10 are 8,2418773675; 6,4637261171; and 4,6855748667. One grade $=$ 0,0157079632679; its logarithm increased by $10 = 8,1961198769$; from which the values for a grade-minute and grade-second are immediately found. The number of degrees contained in the radius is 57,29577; the number of grades is 63,66197. The value of the semi-circumference to radius 1 is generally denoted by π; $\frac{\pi}{2}$ is therefore the value of the quadrant, and $2\,\pi$ that of the circumference.

(5.) The defect of an arc from 180° is called its *supplement;* its defect from 90° is called its *complement.*

(6.) Join A B (fig. 2); draw B D and C F perpendicular to A C; at A and F draw lines touching the circle, which will therefore be parallel to C F, C A; produce C B to cut these lines in E and G. Then A B is the *chord* of the arc A B, B D is the *sine*, C D is the *cosine*, A E is the *tangent*, C E is the *secant*, F G is the *cotangent*, C G the *cosecant*, A D the *versed sine.* D H has been called by some the *suversed sine.*

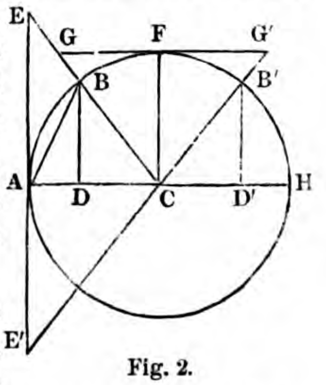

Fig. 2.

(7.) These definitions suppose the arc to be less than a quadrant. If it be greater than a quadrant and less than a semicircle, as A B′, the same construction gives for the sine, versed sine, cosecant, cosine, tangent, secant, and cotangent, the lines B′ D′, A D′, C G′, C D′, A E′, C E′, F G′. The four last of these, it will be observed, are measured in directions opposite to those in which the corresponding lines for arcs less than a quadrant were measured, and are therefore considered negative.* We shall show that, by this convention, formulæ which have been found to be true for arcs less than a quadrant, may be made to apply to arcs greater than a quadrant.

(8.) If the arc be greater than two quadrants, and less than three, as A F H B″ (fig. 3), making the same construction, we find that the sine, cosine, secant, and cosecant are negative. And if the arc be greater than three quadrants, and less than four, as A F H B‴, it appears that the sine, tangent, cotangent, and cosecant are negative.

* The secant is negative, because it is not measured from the centre, in the direction of the radius, through the extremity of the arc, but in the opposite direction.

The remark at the end of (7) applies to these. The versed sine and suversed sine are positive for all values of the arc.

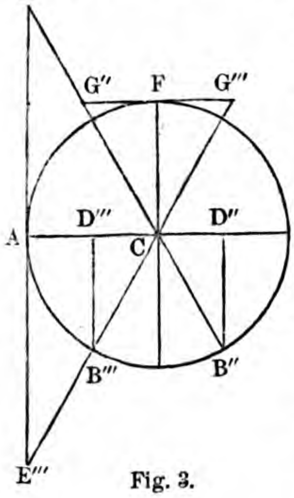

Fig. 3.

(9.) Thus it appears that, while the arc increases from 0 to a quadrant, the sine increases from 0 to radius (its greatest value,) and the cosine diminishes from radius (its greatest value) to 0. While the arc increases to a semicircle, the sine diminishes to 0; and the cosine, whose sign is now negative, increases in magnitude till it $= -$ radius. As the arc increases to three quadrants, the sine is negative, and its magnitude increases from 0 till it $= -$ radius; while the negative value of the cosine diminishes till it $= 0$. From three quadrants to four, the sine, still negative, diminishes its negative value till it $= 0$; while the cosine, become positive, increases till it is $=$ radius, as at first.

(10.) The tangent, while the arc increases from 0 till it is $\frac{\pi}{2}$, increases so as to become greater than any assigned quantity; when the arc $= \frac{\pi}{2}$, or $\frac{3\,\pi}{2}$, there is really no tangent, as the lines, by whose intersection the tangent is defined, do not meet; then, until the arc $= \pi$, the tangent is negative, and diminishes from a value indefinitely great to 0; then, for the third and fourth quadrants the values are the same as for the first and second. And the secant, while the arc increases from 0 to $\frac{\pi}{2}$, increases from radius to a value greater than any assignable; it then becomes negative, and diminishes from a value indefinitely great to radius, which it reaches when the arc $= \pi$; for the third and fourth quadrants its values are the same as for the first and second, with the sign changed.

(11.) If the arc, instead of being $=$ A B, were $=$ A B increased by any number of whole circumferences, the values of the several trigonometrical lines would be the same as those for the arc A B.

(12.) The definitions of the complement and supplement, without some extension, will not apply to arcs greater than 90° or 180° respectively. It is only necessary to consider the defect of the arc from 90° or 180° as being negative when the arc is greater than either of those values; and all the theorems relating to these defects will be comprehended under the same formula.

(13.) Since we have considered positive arcs as measured from A towards F, we may consider negative arcs as measured in the opposite direction. Let A B, A B′ (fig. 4) be equal arcs, positive and negative; their sines B D, B′ D will evidently be in the same straight line; A E′ = A E, F G′ = F G, C E′ = C E, C G′ = C G. Hence, for a negative arc, the cosine, versed sine, and secant, are the same as those for an equal positive arc; the sine, tangent, cotangent, and cosecant are equal in respect of magnitude, but are affected with different signs. Our figure supposes A B less than a quadrant, but it will be seen that the same is true if A B be greater than a quadrant.

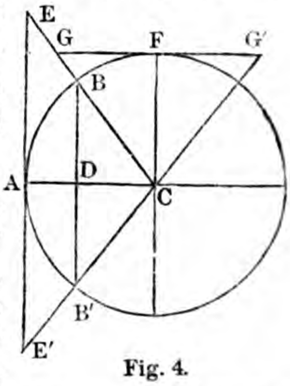

Fig. 4.

(14.) The whole of what we have assumed with regard to the signs to be affixed to the expressions for lines according to their directions, is purely arbitrary. Its utility is this:—A single formula, as we shall show by induction, will now comprehend several cases for which as many separate formulæ would otherwise have been necessary. This, we conceive, is in all cases the true foundation for the use of the negative sign.

Examples to Section I.

1. Convert $25^g\ 36'\ 27''$ into degrees, minutes, and seconds.
2. Find the length in inches of the arc of $12°\ 15'\ 10''$, when the radius of the circle is 8 feet.
3. In the same circle, what is the number of degrees, minutes, and seconds in an arc of the length of a foot.
4. Show how the cotangent, cosecant, and versed sine vary in sign and magnitude, as the arc increases from 0 to 2π.
5. Show, in each case, that the sign of the cosine, cotangent, or cosecant is the same as is given by considering them, respectively, as the sine, tangent, or secant of the complement.
6. Express the sines, &c., of the arcs $\frac{\pi}{2} + A$, $\pi + A$, $\frac{3\pi}{2} + A$, $2\pi - A$, by means of the sine, cosine, &c., of the arc A.

SECTION II.

RELATIONS OF TRIGONOMETRICAL LINES.

(15.) In the succeeding articles we shall use the abbreviations Sin, Cos, Tan, Sec, Cot, Cosec, Vers, to denote the sine, cosine, &c., to the radius R; and sin, cos, &c., to denote them supposing the radius $= 1$.

(16.) If C K L be drawn perpendicular to A B (fig. 5,) A K $=$ K B, the angle A C K $=$ B C K, and the arc A L $=$ B L, therefore A B $= 2 .$ A K. But A K is evidently the sine of A L, or $\frac{1}{2}$ A B. And the straight line A B is the chord of the arc A B. Hence

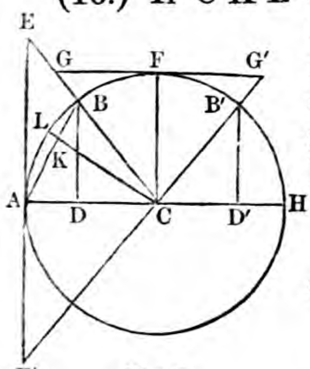

Fig. 5.

$$\text{Chord } AB = 2 . \text{Sin } \frac{AB}{2},$$

and

$$\text{chord } AB = 2 \sin \frac{AB}{2}.$$

(17.) A D $=$ A C $-$ C D, or

$$\text{Vers } AB = R - \text{Cos } AB,$$

and therefore

$$\text{vers } AB = 1 - \cos AB.$$

By the convention established with regard to signs, it will be found that this equation applies to arcs terminated in all quadrants of the circle.

(18.) By similar triangles (*Euclid*, vi. 4,) $AE = \frac{DB \times CA}{CD}$,

or

$$\text{Tan } AB = \frac{R . \text{Sin } AB}{\text{Cos } AB},$$

and

$$\tan AB = \frac{\sin AB}{\cos AB}.$$

(19.) By similar triangles, $FG = \frac{CD \times CF}{DB}$, or

$$\text{Cot } AB = \frac{R . \text{Cos } AB}{\text{Sin } AB},$$

and

$$\cot AB = \frac{\cos AB}{\sin AB}.$$

(20.) Multiplying together these expressions,

$$\text{Tan } AB \times \text{Cot } AB = R^2,$$

and

$$\tan AB . \cot AB = 1.$$

(21.) By similar triangles, $CE = \frac{CA \times CB}{CD}$, or

$$\text{Sec } AB = \frac{R^2}{\text{Cos } AB},$$

and

$$\sec AB = \frac{1}{\cos AB}.$$

(22.) By similar triangles, $CG = \frac{CF \times CB}{DB}$, or

$$\text{Cosec } AB = \frac{R^2}{\text{Sin } AB},$$

and

$$\text{cosec } AB = \frac{1}{\sin AB}.$$

(23.) Suppose $HB' = AB$; then AB', or $180° - HB'$, is the supplement of AB. And $B'D' = BD$, $CD' = CD$, $AE' = AE$, $CE' = CE$, $FG' = FG$, $CG' = CG$, $AD = HD'$. Hence the sine and cosecant of any arc are the same as those of its supplement; the cosine, tangent, cotangent, and secant are equal in magnitude, with different signs; and the versed sine of one is the suversed sine of the other.

(24.) If $Ab = FB$ (fig. 6,) and bd, Ceg, be drawn as before, it is plain that $bd = CD$, $Cd = BD$, $Ae = FG$, $Fg = AE$, $Ce = CG$, $Cg = CE$. But bd, Cd, Ae, Fg, Ce, Cg, are the sine, cosine, tangent, cotangent, secant, and cosecant of Ab or BF; and BF is the complement of AB. Hence the sine, cosine, tangent, cotangent, secant, and cosecant of the complement of an arc, are respectively equal to the cosine, sine, cotangent, tangent, cosecant, and secant of the arc.

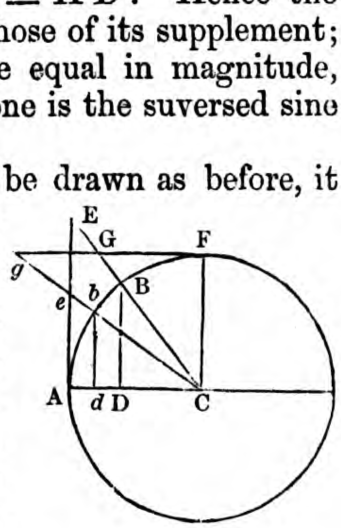

Fig. 6.

(25.) All these theorems have been proved for arcs less than a quadrant. If, however, we make use of the convention established with regard to signs, it will be found that they apply to every case. For example, when the arc, as $AFHB'''$, fig. 3, is greater than three quadrants, and less than four, the sine is negative, the cosine is positive; therefore the tangent $= \frac{\text{sine}}{\text{cosine}}$ (18) ought by the formula to be negative; which, from the figure, it appears to be. The magnitude is determined by the same proportion as before, and cannot be erroneous. The secant $= \frac{1}{\text{cosine}}$ (21) ought to be positive; and the cosecant $= \frac{1}{\text{sine}}$ (19) ought to be negative; as they are found to be. The same, it will be found, is true for every other case.

(26.) By similar triangles the following proportions will easily be verified. Radius : Sin A B :: Sec A B : Tan A B; therefore

$$\text{Sin A B} = \frac{\text{R . Tan A B}}{\text{Sec A B}},$$

and
$$\sin \text{A B} = \frac{\tan \text{A B}}{\sec \text{A B}}.$$

Radius : Cos A B :: Cosec A B :: Cot A B; therefore

$$\text{Cos A B} = \frac{\text{R . Cot A B}}{\text{Cosec A B}},$$

and
$$\cos \text{A B} = \frac{\cot \text{A B}}{\text{cosec A B}}.$$

(27.) Since $(\text{Sec A B})^2 = \text{R}^2 + (\text{Tan A B})^2$ (*Euclid*, i. 47,) or $\sec^2 \text{AB} = 1 + \tan^2 \text{AB}$, and $\text{cosec}^2 \text{AB} = 1 + \cot^2 \text{AB}$, we may thus express these values:

$$\sin \text{A B} = \frac{\tan \text{A B}}{\sqrt{(1 + \tan^2 \text{A B})}} = \frac{\sqrt{(\sec^2 \text{A B} - 1)}}{\sec \text{A B}};$$

$$\cos \text{A B} = \frac{\cot \text{A B}}{\sqrt{(1 + \cot^2 \text{A B})}} = \frac{\sqrt{(\text{cosec}^2 \text{A B} - 1)}}{\text{cosec A B}}.$$

And the equations of (21) and (22) may be thus expressed:

$$\cos \text{A B} = \frac{1}{\sqrt{(1 + \tan^2 \text{A B})}};$$

$$\sin \text{A B} = \frac{1}{\sqrt{(1 + \cot^2 \text{A B})}}.$$

(28.) In the same way, observing that $\sin^2 \text{A B} + \cos^2 \text{A B} = 1$, we find from (18) and (19),

$$\tan \text{A B} = \frac{\sin \text{A B}}{\sqrt{(1 - \sin^2 \text{A B})}} = \frac{\sqrt{(1 - \cos^2 \text{A B})}}{\cos \text{A B}};$$

$$\cot \text{A B} = \frac{\cos \text{A B}}{\sqrt{(1 - \cos^2 \text{A B})}} = \frac{\sqrt{(1 - \sin^2 \text{AB})}}{\sin \text{A B}}.$$

These are the principal formulæ of the relations of trigonometrical lines belonging to one arc.

(29.) We proceed to one of the most important propositions of trigonometry. To find the sine and cosine of the sum and difference of two arcs in terms of the sine and cosine of the simple arcs. Let A B (fig. 7) be the longer arc = A; B E = B F = B; then A E = A + B, A F = A — B. Draw E G, F G, perpendicular to C B, which will meet at G and be in the same straight line, and will be equal; also draw B D, E H, F K, G L, perpendicular to A C; and G M, F N, perpendicular to E H, G L.

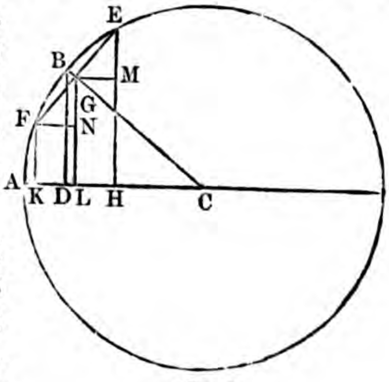

Fig. 7.

Then E H or G L + E M = sin (A + B);
F K or G L — G N = sin (A — B);
C H or C L — G M = cos (A + B);
C K or C L + F N = cos (A — B).

Now the angle E G M = 90° — M G C = C G L = C B D; also E M G and C D B are right angles, therefore the triangles E G M, B C D, are similar; and C B : C D :: E G : E,M

or Radius : Cos A :: Sin B : E M $= \frac{\text{Cos A . Sin B}}{\text{R}} =$ G N.

And C B : B D :: E G : G M,

or Radius : Sin A :: Sin B : G M $= \frac{\text{Sin A . Sin B}}{\text{R}} =$ F N.

Also C B : B D :: C G : G L $= \frac{\text{B D . C G}}{\text{C B}} = \frac{\text{Sin A . Cos B}}{\text{R}}$;

and C B : C D :: C G : C L $= \frac{\text{C D . C G}}{\text{C B}} = \frac{\text{Cos A . Cos B}}{\text{R}}$.

Substituting these values,

$$\text{Sin (A + B)} = \frac{\text{Sin A . Cos B + Cos A . Sin B}}{\text{R}};$$

$$\text{Sin (A — B)} = \frac{\text{Sin A . Cos B — Cos A . Sin B}}{\text{R}};$$

$$\text{Cos (A + B)} = \frac{\text{Cos A . Cos B — Sin A . Sin B}}{\text{R}};$$

$$\text{Cos (A — B)} = \frac{\text{Cos A . Cos B + Sin A . Sin B}}{\text{R}}.$$

Or, if the radius be the unit of measure,

sin (A + B) = sin A . cos B + cos A . sin B;
sin (A — B) = sin A . cos B — cos A . sin B;
cos (A + B) = cos A . cos B — sin A . sin B;
cos (A — B) = cos A . cos B + sin A . sin B.

(30.) It is here supposed that A is greater than B, and that A is less than 90°. If these conditions should not hold, it would still be found that, by virtue of our conventions with regard to the signs of arcs and straight lines, the same formulæ would apply. We shall leave it to the reader to examine in this manner every distinct case,

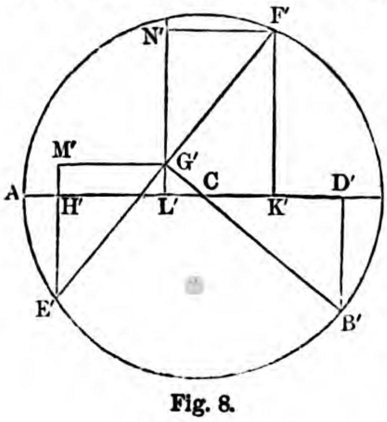

Fig. 8.

and shall, merely as an example, suppose A greater than 180°, B greater than 90°. Let A F' B' (fig. 8) = A; B' E' = B' F' = B. Make the same construction in every respect as before; then $E'H' = E'M' - G'L' = \frac{CD' . E'G'}{CB'} - \frac{B'D' . CG'}{CB'}$.

But, by (7) and (8), since A F' B' E' = A + B, E' H' is = — Sin (A + B); C D' = — Cos A; E' G' = Sin B; B' D' = — Sin A; C G' = — Cos B; thus the equation becomes $-\text{Sin}(A+B) = \frac{-\text{Cos}\,A . \text{Sin}\,B - \text{Sin}\,A . \text{Cos}\,B}{R}$,

or $\text{Sin}(A+B) = \frac{\text{Sin}\,A . \text{Cos}\,B + \text{Cos}\,A . \text{Sin}\,B}{R}$; the same as for arcs less than 90°. And the same will be found to be true for every different case.

(31.) From these expressions,

$$\sin(A+B) + \sin(A-B) = 2 \sin A . \cos B,$$
$$\sin(A+B) - \sin(A-B) = 2 \cos A . \sin B,$$
$$\cos(A+B) + \cos(A-B) = 2 \cos A . \cos B,$$
$$\cos(A-B) - \cos(A+B) = 2 \sin A . \sin B.$$

(32.) Let A + B = C; A — B = D; then

$$\sin C + \sin D = 2 \sin \frac{C+D}{2} . \cos \frac{C-D}{2},$$
$$\sin C - \sin D = 2 \cos \frac{C+D}{2} . \sin \frac{C-D}{2},$$
$$\cos C + \cos D = 2 \cos \frac{C+D}{2} . \cos \frac{C-D}{2},$$
$$\cos D - \cos C = 2 \sin \frac{C+D}{2} . \sin \frac{C-D}{2}.$$

(33.) Let B = A; then sin 2 A = 2 sin A . cos A; and cos 2 A = $\cos^2 A - \sin^2 A = 1 - 2\sin^2 A = 2\cos^2 A - 1$. From these,

$$\sin A = \sqrt{\frac{1-\cos 2A}{2}} = \sqrt{\frac{\text{versin}\,2A}{2}};$$

$$\cos A = \sqrt{\frac{1+\cos 2A}{2}}.$$

If in these values we put $\sqrt{1-\sin^2 2\text{ A}}$ for cos 2 A,

$$\sin \text{A} = \sqrt{\left(\frac{1-\sqrt{1-\sin^2 2\text{ A}}}{2}\right)}$$

$$= \tfrac{1}{2}\left(\sqrt{1+\sin 2\text{ A}} - \sqrt{1-\sin 2\text{ A}}\right);$$

$$\cos \text{A} = \tfrac{1}{2}\left(\sqrt{1+\sin 2\text{ A}} + \sqrt{1-\sin 2\text{ A}}\right).$$

(34.) Again, $\cot \text{A} + \tan \text{A} = \frac{\cos \text{A}}{\sin \text{A}} + \frac{\sin \text{A}}{\cos \text{A}} = \frac{\cos^2 \text{A} + \sin^2 \text{A}}{\sin \text{A} \cos \text{A}}$ $= \frac{1}{\sin \text{A} . \cos \text{A}} = \frac{2}{2 \sin \text{A} . \cos \text{A}} = \frac{2}{\sin 2\text{ A}} = 2 \operatorname{cosec} 2\text{ A}$. Similarly, $\cot \text{A} - \tan \text{A} = \frac{\cos^2 \text{A} - \sin^2 \text{A}}{\sin \text{A} . \cos \text{A}} = \frac{2 \cos 2\text{ A}}{\sin 2\text{ A}} = 2 \cot 2\text{ A}$.

(35.) Since $\sin \text{A} = \sqrt{\frac{1-\cos 2\text{ A}}{2}}$, and $\cos \text{A} = \sqrt{\frac{1+\cos 2\text{ A}}{2}}$, we have

$$\tan \text{A} = \frac{\sin \text{A}}{\cos \text{A}} = \sqrt{\frac{1-\cos 2\text{ A}}{1+\cos 2\text{ A}}}.$$

Hence

$$\cos 2\text{ A} = \frac{1-\tan^2 \text{A}}{1+\tan^2 \text{A}}.$$

(36.) Since $\sin \text{A} = \frac{\tan \text{A}}{\sqrt{(1+\tan^2 \text{A})}}$, and $\cos \text{A} = \frac{1}{\sqrt{(1+\tan^2 \text{A})}}$

by (27), $\sin 2\text{ A} = 2 \sin \text{A} . \cos \text{A} = \frac{2 \tan \text{A}}{1+\tan^2 \text{A}}$.

(37.) $\frac{1-\cos 2\text{ A}}{\sin 2\text{ A}} = \frac{2 \sin^2 \text{A}}{2 \sin \text{A} . \cos \text{A}} = \frac{\sin \text{A}}{\cos \text{A}} = \tan \text{A}$.

Similarly, $\frac{\sin 2\text{ A}}{1+\cos 2\text{ A}} = \tan \text{A}$.

(38.) From (31,) $\sin(\text{A}+\text{B}) = 2 \sin \text{A} . \cos \text{B} - \sin(\text{A}-\text{B}$.) Let $\text{A} = n\text{ B}$; then $\sin(n+1)\text{ B} = 2 \sin n\text{ B} . \cos \text{B} - \sin(n-1)\text{ B}$. Making n successively $= 2, 3$, &c., we form the following table:

$$\begin{aligned}
\sin \text{B} &= \sin \text{B}, \\
\sin 2\text{ B} &= 2 \sin \text{B} . \cos \text{B}, \\
\sin 3\text{ B} &= 3 \sin \text{B} - 4 \sin^3 \text{B}, \\
\sin 4\text{ B} &= (4 \sin \text{B} - 8 \sin^3 \text{B}) \cos \text{B}, \\
\sin 5\text{ B} &= 5 \sin \text{B} - 20 \sin^3 \text{B} + 16 \sin^5 \text{B}, \\
\&\text{c.} & \quad \&\text{c.}
\end{aligned}$$

Again, from (31,) $\cos(\text{A}+\text{B}) = 2 \cos \text{A} . \cos \text{B} - \cos(\text{A}-\text{B}$.) Let $\text{A} = n\text{ B}$, then $\cos(n+1)\text{ B} = 2 \cos n\text{ B} . \cos \text{B} - \cos(n-1)\text{ B}$. Making n successively $= 2, 3$, &c.

$$\begin{aligned}
\cos\ B &= \cos B, \\
\cos 2B &= 2\cos^2 B - 1, \\
\cos 3B &= 4\cos^3 B - 3\cos B, \\
\cos 4B &= 8\cos^4 B - 8\cos^2 B + 1, \\
\cos 5B &= 16\cos^5 B - 20\cos^3 B + 5\cos B, \\
\&c. &\qquad \&c.
\end{aligned}$$

(39.) By putting $A + B$ for A, and $A - B$ for B in (31,)

$$\begin{aligned}
\sin(A+B).\sin(A-B) &= \tfrac{1}{2}(\cos 2B - \cos 2A) \\
&= \tfrac{1}{2}(1 - 2\sin^2 B - 1 + 2\sin^2 A) \text{ by (33)}; \\
&= \sin^2 A - \sin^2 B, \text{ or } = \cos^2 B - \cos^2 A.
\end{aligned}$$

$$\begin{aligned}
\text{And } \cos(A+B).\cos(A-B) &= \tfrac{1}{2}(\cos 2B + \cos 2A) \\
&= \tfrac{1}{2}(1 - 2\sin^2 B + 2\cos^2 A - 1) \\
&= \cos^2 A - \sin^2 B, \text{ or} = \cos^2 B - \sin^2 A.
\end{aligned}$$

(40.) $\dfrac{\sin(A+B)}{\sin(A-B)} = \dfrac{\sin A.\cos B + \cos A.\sin B}{\sin A.\cos B - \cos A.\sin B} = \dfrac{\tan A + \tan B}{\tan A - \tan B}$,

or $= \dfrac{\cot B + \cot A}{\cot B - \cot A}$; and similarly, $\dfrac{\cos(A+B)}{\cos(A-B)} = \dfrac{\cot B - \tan A}{\cot B + \tan A}$,

or $= \dfrac{\cot A - \tan B}{\cot A + \tan B}$.

(41.) $\dfrac{\sin A + \sin B}{\sin A - \sin B} = \dfrac{2\sin\dfrac{A+B}{2}.\cos\dfrac{A-B}{2}}{2\cos\dfrac{A+B}{2}.\sin\dfrac{A-B}{2}} = \dfrac{\tan\dfrac{A+B}{2}}{\tan\dfrac{A-B}{2}}$;

$$\frac{\cos B + \cos A}{\cos B - \cos A} = \frac{2\cos\dfrac{A+B}{2}.\cos\dfrac{A-B}{2}}{2\sin\dfrac{A+B}{2}.\sin\dfrac{A-B}{2}} = \cot\frac{A+B}{2}.\cot\frac{A-B}{2}.$$

(42.) $\tan A + \tan B = \dfrac{\sin A}{\cos A} + \dfrac{\sin B}{\cos B} = \dfrac{\sin A.\cos B + \cos A.\sin B}{\cos A.\cos B}$

$= \dfrac{\sin(A+B)}{\cos A.\cos B}$. Similarly, $\tan A - \tan B = \dfrac{\sin(A-B)}{\cos A.\cos B}$;

$\cot A + \cot B = \dfrac{\sin(A+B)}{\sin A.\sin B}$; $\cot B - \cot A = \dfrac{\sin(A-B)}{\sin A.\sin B}$.

(43.) To find an expression for the tangent of the sum or difference of two arcs: $\tan(A+B) = \dfrac{\sin(A+B)}{\cos(A+B)} = \dfrac{\sin A.\cos B + \cos A.\sin B}{\cos A.\cos B - \sin A.\sin B}$; which, dividing the numerator and denominator by $\cos A.\cos B$, and observing that $\frac{\sin A}{\cos A} = \tan A$, gives $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \,.\, \tan B}$.

Similarly, $\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \,.\, \tan B}$.

If $B = A$, $\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$.

(44.) Hence, $\tan(A + B + C) = \frac{\tan(A+B) + \tan C}{1 - \tan(A+B) \,.\, \tan C}$

$$= \frac{\tan A + \tan B + \tan B - \tan A \,.\, \tan B \,.\, \tan C}{1 - \tan A \,.\, \tan B - \tan A \,.\, \tan C - \tan B \,.\, \tan C}.$$

$$\text{If } C = B = A,\ \tan 3A = \frac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}.$$

If $A + B + C = \pi$, $\tan(A + B + C) = 0$, (10;) hence in that case we have this remarkable equation,

$$\tan A + \tan B + \tan C = \tan A \,.\, \tan B \,.\, \tan C.$$

(45.) These are the most important relations that subsist generally between different arcs. As there are some which depend upon the numerical expression for the lines belonging to particular arcs, we shall proceed to investigate their values.

(46.) Let B C D (fig. 9) be half a right angle, or $AB = 45° = \frac{\pi}{4}$; then the angle CBD = half a right angle $= BCD$, therefore $BD = CD$, therefore $1 = \sin^2 \frac{\pi}{4} + \cos^2 \frac{\pi}{4}$ $= 2 \sin^2 \frac{\pi}{4}$, therefore $\sin \frac{\pi}{4} = \frac{1}{\sqrt{2}} = \cos \frac{\pi}{4}$; $\tan \frac{\pi}{4} = 1 = \cot \frac{\pi}{4}$; $\sec \frac{\pi}{4} = \sqrt{2} = \operatorname{cosec} \frac{\pi}{4}$.

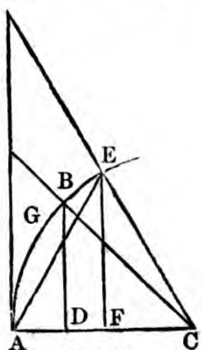

Fig. 9.

(47.) Let $AE = 60° = \frac{\pi}{3}$; then, since the sum of the three angles of the triangle ACE = two right angles $= 180°$, the sum of those at A and $E = 120°$; and, as they are equal, each $= 60° = \frac{\pi}{3}$; therefore the triangle is equilateral, and $CF = AF$

Hence $\cos \frac{\pi}{3} = \frac{1}{2}$; $\sin \frac{\pi}{3} = \sqrt{1 - \frac{1}{4}} = \frac{\sqrt{3}}{2}$; $\tan \frac{\pi}{3} = \sqrt{3}$;

$\cot \frac{\pi}{3} = \frac{1}{\sqrt{3}}$; $\sec \frac{\pi}{3} = 2$; $\operatorname{cosec} \frac{\pi}{3} = \frac{2}{\sqrt{3}}$.

(48.) Let $AG = 36° = 2 \times 18°$; then the complement of AG

$= 54° = 3 \times 18°$; therefore (24), $\sin 2 \times 18° = \cos (3 \times 18°)$, or $2 . \sin 18° \cos 18° = 4 \cos^3 18 - 3 \cos 18°$, by (38); or, dividing by $\cos 18°$, $2 \sin 18° = 4 \cos^2 18° - 3$. Let $\sin 18° = x$; therefore $2x = 1 - 4x^2$, from the solution of which equation,

$$x \text{ or } \sin 18° = \frac{-1 + \sqrt{5}}{4} = \cos 72°;$$

$$\cos 36° = 1 - 2 \sin^2 18° \ (33) = \frac{1 + \sqrt{5}}{4} = \sin 54°;$$

(49.) From these values,

$$\sin (45° + A) = \sin 45° . \cos A + \cos 45° . \sin A = \frac{\cos A + \sin A}{\sqrt{2}};$$

$$\cos (45° + A) = \cos 45° . \cos A - \sin 45° . \sin A = \frac{\cos A - \sin A}{\sqrt{2}};$$

$$\tan (45° + A) = \frac{\tan 45° + \tan A}{1 - \tan 45° . \tan A} = \frac{1 + \tan A}{1 - \tan A};$$

$$\tan (45° - A), \text{ similarly}, = \frac{1 - \tan A}{1 + \tan A};$$

from which $\tan (45° + A) - \tan (45° - A) = \dfrac{4 \tan A}{1 - \tan^2 A} = 2 \tan 2A$ (43). Also, $\sin (60° + A) - \sin (60° - A) = 2 \cos 60° . \sin A = \sin A$. And

$$\sin (72° + A) - \sin (72° - A) = 2 \cos 72° . \sin A = \frac{\sqrt{5} - 1}{2} \sin A.$$

$$\sin (36° + A) - \sin (36° - A) = 2 \cos 36° . \sin A = \frac{\sqrt{5} + 1}{2} \sin A.$$

Subtracting the upper from the lower, and transposing

$$\sin (36° + A) + \sin (72° - A) = \sin A + \sin (36° - A) + \sin (72° + A).$$

If we had taken $\cos (72° + A) + \cos (72° - A)$, &c., we should have found

$$\cos (36° + A) + \cos (36° - A) = \cos A + \cos (72° + A) + \cos (72° - A).$$

(50.) These are the principal formulæ of the arithmetic of sines. Many of them may be proved geometrically, but we have preferred the algebraical investigations, as less cumbrous, and not less satisfactory.

(51.) The values of the trigonometrical lines which have occurred in these theorems (the numerical calculation of which we shall treat of hereafter), for different arcs, have, with their logarithms, been collected in tables. The sines, tangents, &c., themselves are very seldom used, almost all calculations being now conducted by means of their logarithms. With regard to these it is necessary to observe, that the sines and cosines of all arcs, and the tangents of arcs less than

45°, being less than 1, their logarithms are negative; the use of which would be extremely inconvenient. To avoid this, the logarithms of the tables are made greater by 10 than the real logarithms of the numbers; which it is always necessary to keep in mind in using the tables. For instance (using l for the true logarithm, and L for the logarithm of the tables), since $\tan A = \frac{\sin A}{\cos A}$, therefore $l . \tan A = l . \sin A - l . \cos A$, therefore $L . \tan A - 10 = L \sin A - 10 - L \cos A + 10$, or $L \tan A = L \sin A - L \cos A + 10$. The natural sines, &c., are usually given to radius 10000, but upon removing the decimal point four places to the left they are adapted to radius 1.

(52.) In all expressions involving the length of an arc, deduced from operations by the differential calculus, or from series in terms of the sines, &c., radius is supposed to be the unit of measure. To obtain the number of seconds, we must divide the length by 0,000004848137; or add to its logarithm 5,3144251 to find the logarithm of the number of seconds.

Examples to Section II.

1. Show how the equation of (17) applies for values of the arc greater than $\frac{\pi}{2}$.

2. Show that the variations in sign and magnitude of the secant found from the formula (21) agree with those found from geometrical considerations in (10).

3. If cos A be given, sin A may be found from the equation $\sin^2 A + \cos^2 A = 1$, (28). Give the geometrical interpretation of the *two* values of the sine so found.

4. Each of the square roots in the last formula of (33) implies a double sign; explain which of these signs must be taken for different values of the angle A.

5. Apply the equation of (44) to find tan A, when $\tan 3A = 1$. Give the geometrical interpretation of the three roots of the equation.

6. Find cos 18° from *Euclid* iv. 10.

7. Find the sine and cosine of 9° and its multiples.

If A, B, and C are angles of a triangle, prove

8. That $\cos^2 A + \cos^2 B + \cos^2 C + 2 \cos A . \cos B . \cos C = 1$.

9. That $\cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2} = \cot \frac{A}{2} . \cot \frac{B}{2} . \cot \frac{C}{2}$.

10. That $\cot A . \cot B + \cot B . \cot C + \cot C . \cot A = 1$.

11. Prove that $\tan \left(45° + \frac{A}{2}\right) + \cot \left(45° + \frac{A}{2}\right) = 2 \sec A$.

SECTION III.

ON THE USE OF SUBSIDIARY ANGLES.

(53.) THE possession of trigonometrical tables, ready calculated, frequently enables us to shorten very much numerical calculations which have no relation whatever to trigonometry. The angles which are used in this process, being employed simply to expedite a calculation, are called *subsidiary angles*. Their use will be best elucidated by examples.

(54.) Suppose it is wished to calculate $x = \sqrt{a^2 - b^2}$, and suppose that the logarithms of a and b have already occurred in our operations. Here $x = a\sqrt{1 - \frac{b^2}{a^2}}$. If $\frac{b}{a}$ were the sine of an angle θ, x would be $a \times \cos\theta$. Determine θ therefore by the condition $\frac{b}{a} = \sin\theta$, or $\text{L}\sin\theta = \log b + 10 - \log a$ (51), and having found θ in the tables, x will be found from the expression

$$\log x = \log a + \text{L}\cos\theta - 10.$$

(55.) It is required to calculate the expression $x = a \,.\, \cos\phi + b \,.\, \sin\phi$. If we make $\frac{a}{b} = \tan\theta$, this can be put under the form

$$b\,(\tan\theta . \cos\phi + \sin\phi) = \frac{b}{\cos}(\sin\theta . \cos\phi + \cos\theta . \sin\phi) = \frac{b . \sin(\theta + \phi)}{\cos\theta}.$$

Determine θ by the equation $\text{L}\tan\theta = \log a + 10 - \log b$,

and then $\log x = \log b + \text{L}\sin(\theta + \phi) - \text{L}\cos\theta$,

or $\qquad\qquad = \log b + \text{L}\sin(\theta + \phi) + \text{L}\sec\theta - 20$.

(56.) It is required to find the logarithm of $a + b$, the logarithms of a and b being known.

If a and b are of such a nature that both are in all cases positive,

$$\text{make } \frac{b}{a} = \tan^2\theta, \text{ or } 2\,\text{L}\tan\theta = \log b + \log 20 - \log a;$$

then $$a + b = a\left(1 + \frac{b}{a}\right) = a \,.\, \sec^2\theta;$$

and $$\log(a + b) = \log a + 2\,\text{L}\sec\theta - 20.$$

If, however, a and b may be sometimes positive and sometimes negative, the following method must be used:

$$a+b=\sqrt{2}\,.\,\frac{a+b}{\sqrt{2}}=\sqrt{2}\,.\,(a\,.\cos 45^\circ+b\,.\sin 45^\circ).$$

Let $\frac{a}{b}=\tan\theta$, or $\mathrm{L}\tan\theta=\log a+10-\log b$;

then $a+b=\sqrt{2}\,.\,b\,(\tan\theta\,.\cos 45^\circ+\sin 45^\circ)$

$$=\sqrt{2}\,.\,\frac{b}{\cos\theta}(\sin\theta\,.\cos 45^\circ+\cos\theta\,.\sin 45^\circ)$$

$$=\frac{b\sqrt{2}}{\cos\theta}\sin(\theta+45^\circ),$$

and $\log(a+b)=,1505150+\log b+\mathrm{L}\sin(\theta+45^\circ)-\mathrm{L}\cos\theta$.

(**57.**) In physical astronomy the following expression occurs: $\mathrm{P}=(1+e')\,.\,(1+e'')\,.\,(1+e''')\,.\,\&c.$, where

$$e'=\frac{1-\sqrt{1-e^2}}{1-\sqrt{1+e^2}},\quad e''=\frac{1-\sqrt{1-e'^2}}{1+\sqrt{1-e'^2}},\ \&c.$$

Let $e=\sin\theta$;

then $\sqrt{1-e^2}=\cos\theta$; $\frac{1-\sqrt{1-e^2}}{1+\sqrt{1-e^2}}=\frac{1-\cos\theta}{1+\cos\theta}=\tan^2\frac{\theta}{2}$; $1+e'=\sec^2\frac{\theta}{2}$.

Similarly, make e' or $\tan^2\frac{\theta}{2}=\sin\theta'$; and $1+e''=\sec^2\frac{\theta'}{2}$, &c.

Hence, $\log\mathrm{P}=2\left(\mathrm{L}\sec\frac{\theta}{2}+\mathrm{L}\sec\frac{\theta'}{2}+\&c.-10-10-\&c.\right)$

This computation would be almost impracticable in any other way.

(**58.**) The roots of the quadratic $x^2-p\,x-q=0$, being

$$\frac{p}{2}\pm\sqrt{\frac{p^2}{4}+q},\ \text{or}\ \sqrt{q}\,.\left\{\frac{p}{2\sqrt{q}}\pm\sqrt{\frac{p^2}{4\,q}+1}\right\},$$

let $\frac{p}{2\sqrt{q}}=\cot\theta$;

the roots are $\sqrt{q}\,.\,(\cot\theta\pm\operatorname{cosec}\theta)=\sqrt{q}\,.\,\frac{\cos\theta\pm 1}{\sin\theta}$

$$=\text{by (37)}-\sqrt{q}\,.\tan\frac{\theta}{2}\ \text{and}\ \sqrt{q}\,.\cot\frac{\theta}{2}.$$

The roots of $x^2 - p\,x + q = 0$, being $\frac{p}{2}\left(1 \pm \sqrt{1 - \frac{4\,q}{p^2}}\right)$, let $\frac{4\,q}{p^2} = \sin^2\theta$, and the roots are $\frac{p}{2}\,(1 \pm \cos\theta) = p\,.\cos^2\frac{\theta}{2}$ and $p\,.\sin^2\frac{\theta}{2}$.

(**59.**) The possible root of the cubic $x^3 - q\,x - r = 0$ is

$$\sqrt[3]{\left\{\frac{r}{2} + \sqrt{\frac{r^2}{4} - \frac{q^3}{27}}\right\}} + \sqrt[3]{\left\{\frac{r}{2} - \sqrt{\frac{r^2}{4} - \frac{q^3}{27}}\right\}}$$

$$= \sqrt{\frac{q}{3}}\,.\left(\sqrt[3]{\left\{\sqrt{\frac{27\,r^2}{4\,q^3}} + \sqrt{\frac{27\,r^2}{4\,q^3} - 1}\right\}}\right.$$

$$\left. + \sqrt[3]{\left\{\sqrt{\frac{27\,r^2}{4\,q^3}} - \sqrt{\frac{27\,r^2}{4\,q^3} - 1}\right\}}\right).$$

Let $\frac{27\,r^2}{4\,q^3} = \operatorname{cosec}^2\theta$,

$$\text{the root} = \sqrt{\frac{q}{3}}\left\{\sqrt[3]{\frac{1+\cos\theta}{\sin\theta}} + \sqrt[3]{\frac{1-\cos\theta}{\sin\theta}}\right\}$$

$$= \sqrt{\frac{q}{3}}\left\{\sqrt[3]{\cot\frac{\theta}{2}} + \sqrt[3]{\tan\frac{\theta}{2}}\right\}.$$

Let $\sqrt[3]{\tan\frac{\theta}{2}} = \tan\varphi$,

$$\text{the root} = \sqrt{\frac{q}{3}}\,(\cot\varphi + \tan\varphi) = \sqrt{\frac{4\,q}{3}}\operatorname{cosec} 2\,\varphi.$$

If $\frac{q^3}{27}$ be greater than $\frac{r^2}{4}$, let x be assumed $= a\cos\theta$, or $\frac{x}{a} = \cos\theta$;

then $$\cos 3\,\theta = \frac{4\,x^3}{a^3} - \frac{3\,x}{a}, \text{ by (38)},$$

or $$x^3 - \frac{3\,a^2}{4}\,x - \frac{a^3}{4}\cos 3\,\theta = 0;$$

making this coincide with the given equation, $\frac{3\,a^2}{4} = q$, $\frac{a^3}{4}\cos 3\,\theta = r$, which determine a and θ; and $a\cos\theta$, or x, is then immediately found. The equation will also be satisfied by making

$$x = a\,.\cos\left(\theta + \frac{2\,\pi}{3}\right), \text{ or } x = a\,.\cos\left(\theta + \frac{4\,\pi}{3}\right),$$

for these give $x^3 - \frac{3\,a^2}{4}\,x$ equal to $\frac{a^3}{4}\cos(3\,\theta + 2\,\pi)$ and $\frac{a^3}{4}\cos(3\,\theta + 4\,\pi)$, which by (11) are each equal to $\frac{a^3}{4}\cos 3\,\theta$.

Examples to Section III.

1. Find all the values of x which satisfy the equation,

$$a \sin x + b \cos x = c.$$

2. Show how to calculate $x = \sqrt{a+b} + \sqrt{a-b}$ by the help of logarithmic tables.

3. Solve the equations, $x^3 - 3x + 1 = 0.$

$$x^3 - 7x + 6 = 0.$$

SECTION IV.

PLANE TRIGONOMETRY.

(60.) A TRIANGLE consists of six parts, viz., three sides and three angles; and if any three of these be given, the triangle is completely defined. The case must be excepted in which the three angles are given; as then the proportion only of the sides can be found, the absolute magnitudes remaining unknown. To determine in number the values of three parts from those of three given parts, is the special object of plane trigonometry.

(61.) Suppose the triangle right-angled, let a and b be the sides containing the right angle, c the third side, A, B, C the angles opposite (fig. 10). If the hypotenuse and the angle B be given, describe a circle D E to radius 1; draw D F and E G perpendicular to B C; then D F is sin B, B F is cos B, E G is tan B. And AB : BC :: DB : BF, or $c : a :: 1 : \cos B$,

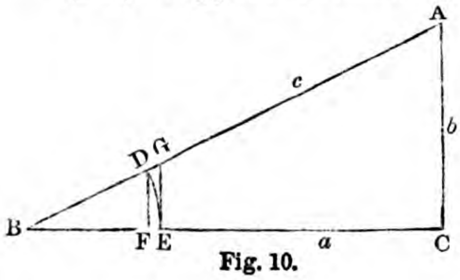

Fig. 10.

therefore $$a = c \cos B.$$

Also A B : A C :: D B : D F, or $c : b :: 1 : \sin B$,

therefore $$b = c \sin B.$$

And the angle $$A = \frac{\pi}{2} - B.$$

(62.) If a and B be given, B C : C A :: B E : E G, or $a : b :: 1 : \tan B$,

therefore $$b = a \tan B.$$

And B C : B A :: B E : B G, or $a : c :: 1 : \sec B$,

therefore $$c = a \sec B.$$

If b and B be given,

$$a = \frac{b}{\tan B} = b \cot B;$$

$$c = \frac{b}{\sin B} = b \operatorname{cosec} B.$$

(63.) If a and c be given,

$$b = \sqrt{(c^2 - a^2)},$$

$$\cos B = \frac{a}{c} = \sin A.$$

If a and b be given,

$$c = \sqrt{(a^2 + b^2)},$$

$$\tan B = \frac{b}{a} = \cot A.$$

(64.) Now, suppose the triangle to be any whatever, we shall first prove this general proposition:—The sides of a triangle are in the same proportion as the sines of the angles opposite. In figs. 11 and

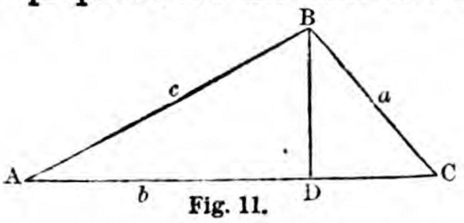

Fig. 11.

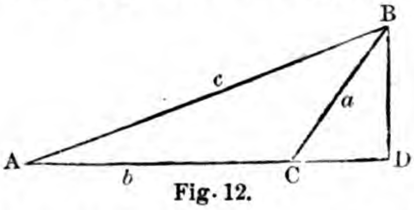

Fig. 12.

12 draw B D a perpendicular from B on A C, or A C produced; then B D = A B sin A (61), and B D also = C B . sin BCA, whether BCA be greater or less than 90° (23); therefore A B . sin A = CB . sin BCA, or

$$AB : CB :: \sin BCA : \sin BAC.$$

(65.) Suppose the three sides of a triangle given, to find the angles. In fig. 11, $BA^2 = BC^2 + CA^2 - 2\,AC\,.\,CD$ (*Euclid* ii. 13); in fig. 12, $BA^2 = BC^2 + CA^2 + 2\,AC\,.\,CD$ (*Euclid* ii. 12). Now in the former case, by (61), $CD = BC\,.\cos C$; in the latter, $CD = BC\,.\cos(\pi - C) = -BC\cos C$ (23); therefore, generally, $AB^2 = BC^2 + CA^2 - 2\,AC\,.\,BC\,.\cos C$, or

$$c^2 = a^2 + b^2 - 2\,ab\,.\cos C.$$

Hence

$$\cos C = \frac{a^2 + b^2 - c^2}{2\,ab}.$$

This formula is very inconvenient for logarithmic computation.

(66.) By (33) we have

$$1 + \cos C, \text{ or } 2\cos^2\frac{C}{2} = \frac{(a+b)^2 - c^2}{2\,ab}$$

$$= \frac{(a+b+c)\,.\,(a+b-c)}{2\,ab} = 2\,\frac{\frac{a+b+c}{2}\,.\left(\frac{a+b+c}{2} - c\right)}{ab}.$$

Let $\frac{a+b+c}{2} = s$, therefore

$$\cos\frac{}{2} = \frac{s\,.\,(s-c)}{ab}.$$

Also (33), $$1 - \cos C, \text{ or } 2\sin^2\frac{C}{2} = \frac{c^2 - (a-b)^2}{2ab}$$

$$= \frac{(c+a-b)\,.\,(c-a+b)}{2ab} = 2\,.\,\frac{(s-b)\,.\,(s-a)}{ab};$$

therefore $$\sin^2\frac{C}{2} = \frac{(s-b)\,.\,(s-a)}{ab}.$$

Dividing this by the last,

$$\tan^2\frac{C}{2} = \frac{(s-b)\,.\,(s-a)}{s\,.\,(s-c)}.$$

Multiplying the product of $\sin^2\frac{C}{2}$ and $\cos^2\frac{C}{2}$ by 4, since $\sin C = 2\sin\frac{C}{2}\cos\frac{C}{2}$ (33),

$$\sin^2 C = \frac{4\,.\,s\,.\,(s-a)\,.\,(s-b)\,.\,(s-c)}{a^2 b^2}.$$

All these expressions, but more particularly the second, are very convenient for the application of logarithms. If two or three angles were required, the formula for $\tan^2\frac{C}{2}$ would probably be most convenient, as the same numbers would be used for the three calculations; or when one angle is found, the theorem of (64) may be applied.

(67.) From the last expression we derive the formula for the area of a triangle in terms of the sides. For

$$\text{the area} = \frac{CA\,.\,BD}{2} = \frac{ab\sin C}{2} = \sqrt{s\,.\,(s-a)\,.\,(s-b)\,.\,(s-c)}.$$

(68.) Suppose now two sides and the angle they contain (a, b, C) to be given, to find the other angles. By (64), $\frac{\sin A}{\sin B} = \frac{a}{b}$,

therefore $$\frac{\sin A + \sin B}{\sin A - \sin B} = \frac{a+b}{a-b},$$

or (41), $$\frac{\tan\frac{A+B}{2}}{\tan\frac{A-B}{2}} = \frac{a+b}{a-b}.$$

Now $A + B + C = \pi$, therefore $\frac{A+B}{2} = \frac{\pi}{2} - \frac{C}{2}$,

therefore, by (24), $\tan\frac{A+B}{2} = \cot\frac{C}{2}$,

and therefore $\tan\frac{A-B}{2} = \frac{a-b}{a+b}\cot\frac{C}{2}$.

When the logarithms of a and b are known, the operation is facilitated thus: Let $\frac{a}{b} = \tan\theta$, therefore

$$\frac{a-b}{a+b} = \frac{\tan\theta - 1}{\tan\theta + 1} = \frac{\tan\theta - \tan 45^\circ}{1 + \tan\theta \,.\, \tan 45^\circ} = \tan(\theta - 45^\circ),$$

and $\tan\frac{A-B}{2} = \tan(\theta - 45^\circ) \,.\, \cot\frac{C}{2}$.

Or thus, if b be less than a, let $\frac{b}{a} = \cos\phi$,

then $\frac{a-b}{a+b} = \frac{1-\cos\phi}{1+\cos\phi} = \tan^2\frac{\phi}{2}$ (35),

and $\tan\frac{A-B}{2} = \tan^2\frac{\phi}{2}\ \cot\frac{C}{2}$

Then $\frac{A+B}{2}$ and $\frac{A-B}{2}$ being known, their sum gives the value of A, and their difference that of B. The third side may be found by the proportion of (64).

(69.) Sometimes, however, it is desirable to find the third side without finding the two remaining angles. In this case, by (65), $c^2 = a^2 + b^2 - 2\,a\,b\cos C = a^2 + 2\,a\,b + b^2 - 2\,a\,b\,(1 + \cos C)$

$$= (a+b)^2 \,.\left\{1 - \frac{4\,a\,b}{(a+b)^2}\cos^2\frac{C}{2}\right\}.$$

Let $\frac{4\,a\,b}{(a+b)^2}\cos^2\frac{C}{2} = \sin^2\theta$; then $c = (a+b) \,.\, \cos\theta$.

Or $c^2 = a^2 - 2\,a\,b + b^2 + 2\,a\,b\,(1 - \cos C)$

$$= (a-b)^2 \,.\left\{1 + \frac{4\,a\,b}{(a-b)^2}\sin^2\frac{C}{2}\right\}.$$

Let $\frac{4\,a\,b}{(a-b)^2}\sin^2\frac{C}{2} = \tan\theta$, then $c = (a-b) \,.\, \sec\theta$.

Or, since $\cos C = \cos^2 \frac{C}{2} - \sin^2 \frac{C}{2}$, by (33), and $1 = \cos^2 \frac{C}{2} + \sin^2 \frac{C}{2}$,

$$c^2 = (a-b)^2 \cdot \cos^2 \frac{C}{2} + (a+b)^2 \cdot \sin^2 \frac{C}{2}$$

$$= (a-b)^2 \cdot \cos^2 \frac{C}{2} \left\{ 1 + \left(\frac{a+b}{a-b}\right)^2 \cdot \tan^2 \frac{C}{2} \right\}.$$

Let $\frac{a+b}{a-b} \tan \frac{C}{2} = \tan \theta$, then $c = (a-b) \cdot \cos \frac{C}{2} \cdot \sec \theta$.

All these are easily calculated by logarithms.

(70.) If two sides and an angle opposite one of them (a, b, A) be given, the angle B is found by the proportion,

$$a : b :: \sin A : \sin B \ (64);$$

then $$C = \pi - A - B,$$

and $$a : c :: \sin A : \sin C.$$

(71.) If c, A, B be given, $C = \pi - A - B$; and the three angles and one side being known, the other sides are easily found by (64).

(72.) If A, B, a be given,

$$C = \pi - A - B, \text{ and } b = \frac{a \cdot \sin B}{\sin A}, c = \frac{a \cdot \sin C}{\sin A}.$$

These forms comprehend all the cases of plane trigonometry.

(73.) In using these formulæ we must, however, observe, that we shall in certain cases arrive at results, the meaning of which is apparently doubtful. These are called the ambiguous cases. We proceed to distinguish those in which the ambiguity is apparent, from those in which it is real.

(74.) First, then, we may observe, that the lengths of lines determined by the formulæ above, since they are the results of simple multiplication and division, and are not given by the solution of quadratic equations, are perfectly free from ambiguity.

(75.) In the next place, an angle when determined by the value of its cosine, versed sine, tangent, cotangent, or secant, is not ambiguous. For the values of the tangent and cotangent, which correspond to the arc A, correspond also to the arc $\pi + A$ (10) and to no smaller arc; the values of the cosine, versed sine, and secant, belong to the arc $2\pi - A$, and to no smaller arc, by (9) and (10); and these being greater than π, or 180°, cannot be used in calculations of triangles.

(76.) But if an angle be determined by the value of its sine or cosecant, since these by (23) belong equally to the arc A and $\pi - A$, both of which, when the sine is positive, are less than π, the value of the arc is apparently doubtful. We will examine every case in which these expressions are found.

(77.) In right-angled triangles the angles must be less than $\frac{\pi}{2}$, and there is therefore no ambiguity. When the angle C in (66) is found by the expression for $\sin \frac{C}{2}$, since C must be less than π, $\frac{C}{2}$ must be less than $\frac{\pi}{2}$, and there is no ambiguity. If found by the expression for sin C, it must be observed that C is greater or less than $\frac{\pi}{2}$, according as c^2 is greater or less than $a^2 + b^2$. In the case of two sides and an angle opposite one being given (70), if a be greater than b, there is no ambiguity; for in the triangle A C B (fig. 13) the angle B must be less than A, and must therefore be less than $\frac{\pi}{2}$, (as if A be greater than $\frac{\pi}{2}$, sin B being less than sin A, of the arcs

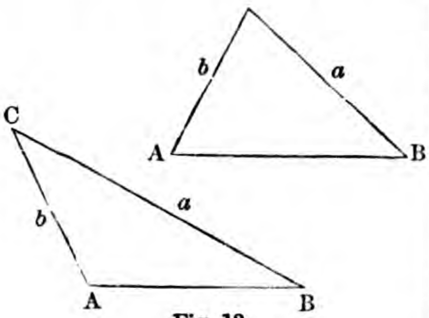

Fig. 13.

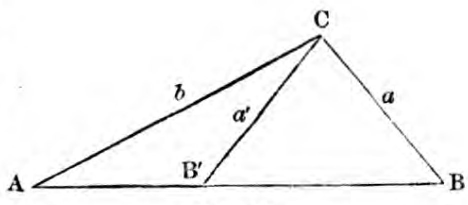

Fig. 14.

corresponding to it one is less than $\frac{\pi}{2}$, the other greater than A). But if a be less than b, the angle A being less than $\frac{\pi}{2}$, (fig. 14), there is nothing to determine whether B is greater or less than $\frac{\pi}{2}$; that is, whether the triangle A C B or A C B′ is to be taken. In this case, then, and in this alone, there is a real ambiguity.

Examples to Section IV.

In a plane triangle

1. Given $\left\{\begin{array}{l} C=90^\circ \\ B=49^\circ\,18' \\ a=1578 \end{array}\right\}$ find $\left\{\begin{array}{l} A=40^\circ\,12' \\ c=2420 \\ b=1834{\cdot}6 \end{array}\right.$

2. Given $\left\{\begin{array}{l} C=90^\circ \\ B=67^\circ\,22'\,48\tfrac{1}{2}'' \\ c=589{\cdot}251 \text{ yards} \end{array}\right\}$ find $\left\{\begin{array}{l} A=22^\circ\,37'\,11\tfrac{1}{2}'' \\ b=543{\cdot}924 \text{ yards} \\ a=226{\cdot}635 \end{array}\right.$

3. Given $\left\{\begin{array}{l} C=90^\circ \\ b=423 \\ a=584 \end{array}\right\}$ find $\left\{\begin{array}{l} A=54^\circ\,5'\,1'' \\ B=35^\circ\,54'\,59'' \\ c=721{\cdot}1 \end{array}\right.$

4. Given $\left\{\begin{array}{l} a=608{\cdot}775 \text{ yards} \\ b=1363{\cdot}656 \text{ yards} \\ c=949{\cdot}689 \text{ yards} \end{array}\right\}$ find $\left\{\begin{array}{l} A=22^\circ\,37'\,11''{\cdot}5 \\ B=120^\circ\,30'\,36''{\cdot}8 \\ C=36^\circ\,52'\,11''{\cdot}7 \end{array}\right.$

Area $= 249047{\cdot}9$ square yards.

5. Given $\left\{\begin{array}{l} a=424{\cdot}096 \text{ yards} \\ b=371{\cdot}084 \text{ yards} \\ C=21^\circ\,47'\,12''{\cdot}5 \end{array}\right\}$ find $\left\{\begin{array}{l} A=98^\circ\,12'\,47''{\cdot}5 \\ B=60^\circ \\ c=159{\cdot}036 \text{ yards.} \end{array}\right.$

6. Given $\left\{\begin{array}{l} A=27^\circ\,47'\,44''{\cdot}8 \\ a=733{\cdot}04 \\ b=837{\cdot}76 \end{array}\right\}$ find $\left\{\begin{array}{l} B=32^\circ\,12'\,15''{\cdot}2 \\ C=120^\circ \\ c=1361{\cdot}36 \end{array}\right.$

or $\left\{\begin{array}{l} B=147^\circ\,47'\,44''{\cdot}8 \\ C=4^\circ\,24'\,30''{\cdot}4 \\ c=120{\cdot}831 \end{array}\right.$

7. Given the angles of a triangle and the radius of the circumscribing circle; find the sides.

8. Given a side and an angle opposite or adjacent, and the sum or the difference of the other sides; find the remaining parts.

9. Given the angles and the area; find the sides of a triangle.

10. Given the angles and the perimeter; find the sides.

11. Given the angles and the radius of the inscribed circle; find the sides.

If R be the radius of the circumscribing circle, r that of the inscribed circle, α, β, γ, those of the circles touching the sides a, b, c, respectively and the other two produced, prove

12. That $r = 4\,\mathrm{R} \sin \frac{1}{2}\mathrm{A} \, . \, \sin \frac{1}{2}\mathrm{B} \, . \, \sin \frac{1}{2}\mathrm{C}$,

and $\alpha = 4\,\mathrm{R} \sin \frac{1}{2}\mathrm{A} \cdot \cos \frac{1}{2}\mathrm{B} \cdot \cos \frac{1}{2}\mathrm{C}$.

13. That $\frac{1}{r} = \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$.

14. That the area $= \sqrt{r \, . \, \alpha \, . \, \beta \, . \, \gamma}$.

15. At a horizontal distance of 115 feet from the bottom of a tower, the angle of elevation of its top was observed to be 51° 30′; find the height.

16. The distance in a straight line between two stations visible from each other is known, show how, with an instrument for measuring angles, to find the height and distance of a hill seen in any direction from both.

17. The distance between two points, A and B, on the banks of a river is measured, and found to be 25 feet 7 inches, find the breadth of the river from A to C on the other side, the angles B A C and A B C being 95° 32′ and 33° 27′ respectively.

18. Calculate the distance between two points, A and B, which are both visible from the points C and D, having found that C D = 60 yards, B D C = 121° 35′, A D C = 49° 20′, A C D = 89° 36′ 35″, B C D = 31° 22′ 30″, and A C B = 67° 15′ 40″. Ans. A B = 106·856.

SECTION V.

SPHERICAL GEOMETRY.

BEFORE proceeding to the solution of spherical triangles, it is necessary to be acquainted with some definitions and propositions in spherical geometry, which are collected in this section.

(78.) A *sphere* is a solid bounded by a surface of which every point is equally distant from a point within it, called the *centre*. A straight line drawn from the centre to the surface, is called a *radius;* if produced both ways to meet the surface, it is a *diameter*.

(79.) Every section of a sphere by a plane is a circle. Let A B (fig. 15) be any section of a sphere made by a plane; from the centre O draw O C perpendicular to this plane; take D, K, any points in the section, and join C D, O D, C K, O K. Since O C is perpendicular to the plane, it is perpendicular to every line which meets it in the plane; therefore OCD, OCK are right angles, and $CD = \sqrt{OD^2 - OC^2}, CK = \sqrt{OK^2 - OC^2}$. But $OK = OD$, therefore $CK = CD$, or the section is a circle of which C is the centre.

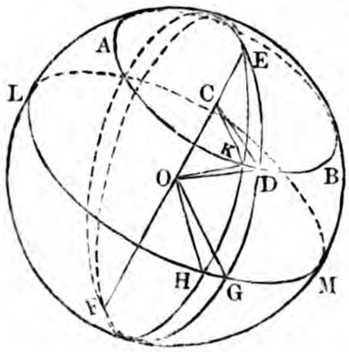

Fig. 15.

(80.) A *great circle* is one whose plane passes through the centre of the sphere; a *small circle* is one whose plane does not pass through the centre. Hence a radius of a great circle is a radius of the sphere. Two circles are said to be parallel when their planes are parallel.

(81.) A great circle may be drawn through any two points on the surface of a sphere, but not generally through more than two. For the plane of a great circle must also pass through the centre of the sphere; and a plane may be made to pass through any three given points, but not generally more than three. A small circle may be drawn through any three given points.

(82.) Two great circles bisect each other. For the intersection of their planes, being a straight line passing through the centre, is a diameter of the sphere, and is therefore a diameter of both circles; and the circles are therefore bisected.

(83.) The *inclination* of two great circles, is the angle made by their tangents at the point of intersection. Since each of these tangents is perpendicular to the radius in which the planes of the circles intersect, the same angle measures the inclination of the planes of the circles, (*Euclid* xi. def. 6).

(84.) If through the centre of a circle, whether great or small, a straight line be drawn perpendicular to its plane, the point in which, if produced, it meets the surface of the sphere, is called the *pole* of that circle. Thus, in fig. 15, F C E being perpendicular to the plane of A B D, and passing through its centre C, E and F are the poles of A D B. From the demonstration of (79) it is evident, that this line will always pass through the centre of the sphere. In a small circle the term *pole* is more usually applied to that point only, as E, which is nearest to the circle.

(85.) If a great circle be made to pass through D and E, and another through K and E, and if the chords D E, K E be drawn; then, since C D is equal to C K, and C E is common, and the angle E C D = E C K, both being right angles, the chord E D is equal to the chord E K, and the arc E D = arc E K. Hence the pole of a circle is equally distant from every point of that circle; the distances being measured by arcs of great circles.

(86.) If E be the pole of the great circle G H, since the centre of this circle is the same with the centre of the sphere, E O G is a right angle, and E G is a quadrant. The distance, therefore, of every point of a great circle from its pole is a quadrant of a great circle. Since E O is perpendicular to G O H, the plane E O G is perpendicular to the plane G O H, and the angle E G H is therefore, by (83), a right angle. And the tangent of G M at G is perpendicular to the tangent of G E; and it is also perpendicular to G O, therefore it is perpendicular to the plane E O G (*Euclid* xi. 4); so also is the tangent of D B at D, which is parallel to it (*Euclid* xi. 8); therefore the tangent of D B at D is perpendicular to the tangent of D E.

(87.) The inclination of E G, E H, which is measured by the inclination of their tangents at E, since these tangents are parallel to O G and O H respectively, is also measured by the angle G O H, or the arc G H.

(88.) Since a line which is perpendicular to two lines meeting it in a plane is perpendicular to that plane, if a point E can be found such that its distance, measured by a great circle, from each of two points G and H not in the same diameter, is a quadrant, that point is the pole of the great circle passing through G and H.

(89.) If in a plane perpendicular to another plane a line be drawn at right angles to their common intersection, it will be perpendicular to the second plane (*Euclid* xi. def. 4). Hence, if G E be drawn, so that E G H is a right angle, and G E be made = a quadrant, E will be the pole of the circle L G M.

(90.) If D K be a small circle parallel to G H, the line O C is perpendicular to both their planes, and therefore, by (84), E is the pole of both. And the angle D C K is equal to the angle G O H. Hence D K, the part of the small circle A B intercepted between the two great circles E D G, E K H, passing through their common pole : H G, the part of the great circle L M intercepted in the same manner : : C D : O G : : sin E D : radius. If the radius of the sphere = 1, this ratio becomes sin E D : 1.

(91.) A *spherical triangle* is a portion of the surface of a sphere contained by three arcs of great circles.

(92.) Any two sides of a spherical triangle taken together are greater than the third. For the arcs A B, B C, C A (fig. 16), being arcs of circles whose radii are equal, are measures of the angles A O B, B O C, C O A, at the centre; and when a solid angle is formed by three plane angles, any two of these taken together are greater than the third (*Euclid* xi. 20); hence any two of the sides A B, B C, C A, taken together, are greater than the third.

(93.) The sum of the three angles A O B, B O C, C O A, is less than four right angles (*Euclid* xi. 21); and, consequently, the sum of the sides A B, B C, C A, is less than a whole circumference, or 2π.

(94.) The surface of the sphere included between E G F, E H F (fig. 15), is proportional to the angle H E G. For if the angle H E G be repeated any number of times, it is quite evident that the area will be repeated as often, and therefore the whole area will be proportional to the number of the repetitions, or to the whole angle. Hence the area E H F G E is to the whole surface as H E G is to four right angles, or 2π. Now the surface of a sphere whose radius is r is $4\pi . r^2$; hence the surface E H F G E $= 2 r^2 \times$ H E G.

(95.) Produce all the sides of the spherical triangle ABC (fig. 16), so as to form complete circles; let D, E, F, be the points of their intersections. Now, (82) the arc A D = semicircle = C F, therefore A C = D F. Similarly, A B = D E, B C = E F. And the angle at A = the angle at D, since (83) each of these is the same as the inclination of the planes A B D, A C D; similarly, the angles at B and C are equal to those at E and F respectively. Hence the triangle A B C is in every respect similar and equal to D E F, and therefore encloses an equal surface. Similarly, A F E = B D C, B F D = A E C. Let the area of A B C or D E F $= x$; that of B D C or A F E = P; that of A E C or B F D = Q;

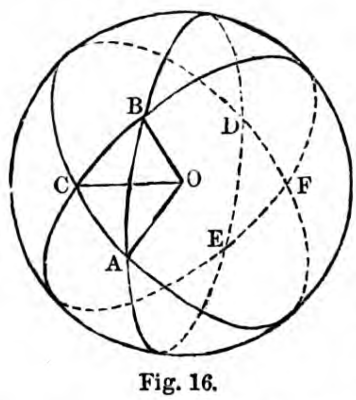

Fig. 16.

that of $AFB = R$. Then, by (94), since x and P together make up the space included by ABD, ACD, we have

$$x + P = 2r^2 \times A.$$

Similarly,
$$x + Q = 2r^2 \times B.$$
$$x + R = 2r^2 \times C.$$

(A being taken to represent the arc corresponding to the angle at A, to radius 1.)

Adding them, $2x + x + P + Q + R = 2r^2 \times (A + B + C)$. But $x + P + Q + R = EDF + AFE + BDF + BFA$ = area defined by $BDEA$ = surface of hemisphere $= 2\pi r^2$, therefore $2x + 2\pi r^2 = 2r^2 (A + B + C)$, therefore $x = r^2 (A + B + C - \pi)$. If $r = 1$, $x = A + B + C - \pi$. The area of a spherical triangle, therefore, is proportional to the excess of the sum of its angles above two right angles. This is usually called the *spherical excess.*

(96.) Suppose great circles EF, FD, DE (fig. 17), to be described, of which A, B, C are respectively the poles; they will intersect in points D, E, F, and form a spherical triangle, called the *polar* or *supplemental triangle.* Now, since A is the pole of EF, the arc joining A and F is a quadrant, by (86); since B is the pole DF, the arc joining B and F is also a quadrant; hence F is the pole of AB (88). Similarly, D and E are the poles of BC, AC, and therefore the triangle ABC is the polar triangle of DEF.

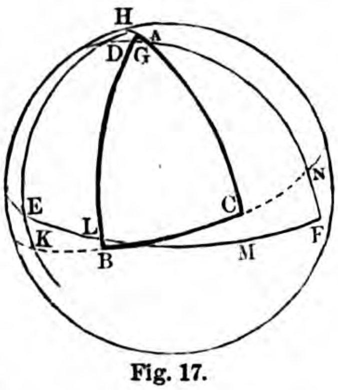

Fig. 17.

(97.) Produce the sides of ABC, if necessary, to meet the sides of the polar triangle. Now, D being the pole of KBC, DK = quadrant; similarly, EH = quadrant, therefore DE = DK + EH — HK = semicircle — HK. But as CH and CK are each = a quadrant, HK is the measure of the angle at C, by (87); hence the sides of the polar triangle are supplements of the angles of the original triangle. Similarly, since the relation between the triangles is reciprocal, the angles of the polar triangle are supplements of the sides of the original triangle.

(98.) The sum of the sides of the polar triangle and the angles of the original triangle $= 3\pi$. Now, the sides of the polar triangle must have some magnitude, and their sum (93) is less than 2π; hence the sum of the angles of the original triangle must be less than 3π, and greater than π.

(99.) A *right-angled spherical triangle* is a spherical triangle having at least one of its angles a right angle.

(100.) If we describe the polar triangle corresponding to a right-angled triangle, one at least of its sides will $= \frac{\pi}{2}$, (97). This is called a *quadrantal triangle.*

(101.) Let A B C be a triangle right-angled at C, (fig. 18); produce the sides A B, C B, to D and E, making $AD = CE = \frac{\pi}{2}$; join E D, and produce it to meet A C produced in F; E B D is called the *complemental triangle.* Since $EC = \frac{\pi}{2}$, and A C E is a right angle, E is the pole of A C, and $EA = EF = \frac{\pi}{2}$, by (89) and (86). And because $AE = AD = \frac{\pi}{2}$, A is the pole of E D, and $AF = \frac{\pi}{2}$. Since A F and A D each $= \frac{\pi}{2}$, D F measures the angle A, (87). But E D is the complement of D F, therefore E D is the complement of A. Similarly, the angle E is the complement of A C. And the side B D is evidently the complement of the hypotenuse A B. The angle A D E being a right angle, the complemental triangle is also a right-angled triangle.

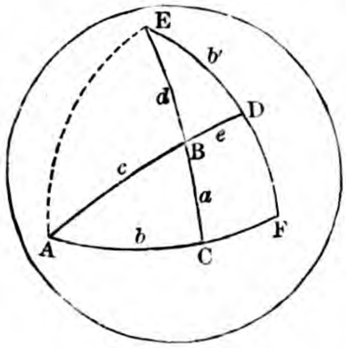

Fig. 18.

SECTION VI.

SPHERICAL TRIGONOMETRY.

(102.) THE sines of the sides of a spherical triangle are proportional to the sines of the opposite angles. Let A B C, (fig. 19), be any spherical triangle: from C draw C D perpendicular on the plane A O B, meeting it in D; and from D draw in that plane D E, D F perpendicular to A O, B O, and join C E, C F, D O. Now, $CE^2 = CD^2 + DE^2 = CO^2 - OD^2 + DE^2$ (since C D being perpendicular to the plane AOB is perpendicular to DE, DO) $= CO^2 - OE^2$; therefore the angle CEO is a right angle, and the angle CED (83) = A, and C E is the sine of A C. Hence $CD = CE \cdot \sin CED = \sin AC \cdot \sin A$. Similarly, $CD = \sin CB \cdot \sin B$. Hence $\sin CA \cdot \sin A = \sin CB \cdot \sin B$, or $\sin CA : \sin CB :: \sin B : \sin A$.

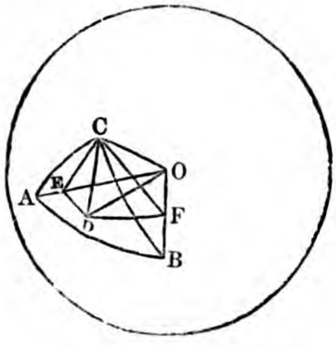

Fig. 19.

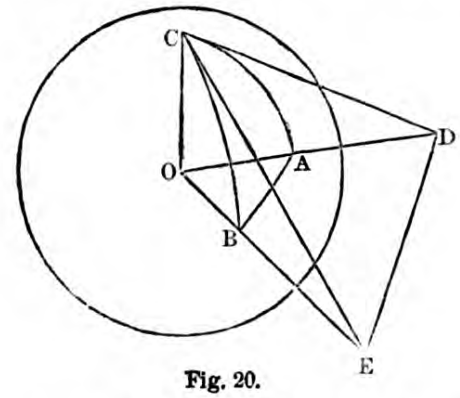

Fig. 20.

(103.) To find the cosine of one angle of a spherical triangle when the three sides are given. Let A B C (fig. 20), be the triangle; draw C D, C E, tangents to C A, C B, and O D, O E secants; join D E. Then (83) the angle made by D C, E C, is the angle C; also, the angle D O E is measured by A B.

Now, $DE^2 = DC^2 + CE^2 - 2\,DC \cdot CE \cdot \cos DCE$, (65),

and $DE^2 = DO^2 + OE^2 - 2\,DO \cdot OE \cdot \cos DOE$.

Comparing these values, and substituting for D C, &c.,

$$\tan^2 AC + \tan^2 BC - 2 \tan AC \cdot \tan BC \cdot \cos C$$
$$= \sec^2 AC + \sec^2 BC - 2 \sec AC \cdot \sec BC \cdot \cos AB.$$

But $\sec^2 AC = 1 + \tan^2 AC, \sec^2 BC = 1 + \tan^2 BC$;
subtracting from both sides $\tan^2 AC + \tan^2 BC$,

$$-2 \tan AC . \tan BC . \cos C = 2 - 2 \sec AC . \sec BC . \cos AB;$$

or $-\dfrac{2 \sin AC . \sin BC . \cos C}{\cos AC . \cos BC} = 2 - \dfrac{2 \cos AB}{\cos AC . \cos BC}$; from which

$$\cos C = \frac{\cos AB - \cos AC . \cos BC}{\sin AC . \sin BC}.$$

It is convenient to denote the sides opposite to the angles A, B, C, by the letters a, b, c; then

$$\cos C = \frac{\cos c - \cos a . \cos b}{\sin a . \sin b}.$$

(104.) This is the fundamental formula of spherical trigonometry: the theorem of (102) may be deduced from it, but as the process is rather long, and as the geometrical proof is very simple, we have preferred establishing it on an independent demonstration. We shall now proceed to investigate the formulæ best adapted for the logarithmic computation of spherical triangles; the general problem being, as in plane trigonometry, from any three given parts (sides or angles) to find the other three. And we shall begin with right-angled triangles.

(105.) Let A B C, (fig. 18), be the triangle, having the angle at C a right angle. By the formulæ (103), $\cos C = \dfrac{\cos c - \cos a . \cos b}{\sin a . \sin b}$; but $C = 90°$, $\cos C = 0$, therefore $\cos c = \cos a . \cos b$.

(106.) Hence in the complemental triangle E B D, which is right-angled, $\cos d = \cos b' . \cos e$; by the relation given in (101) this is immediately transformed into $\sin a = \sin c . \sin A$; similarly, $\sin b = \sin c . \sin B$. This might have been proved by (102).

(107.) Since $\sin b' = \sin d . \sin B$, we have $\cos A = \cos a . \sin B$. And $\cos B = \cos b . \sin A$. Multiplying these equations,

$$\cos B . \cos A = \cos b . \cos a . \sin B . \sin A, \text{ or } \cot A . \cot B = \cos c.$$

(108.) Hence $\cot E . \cot B = \cos d$, or $\tan b . \cot B = \sin a$. Similarly, $\tan a . \cot A = \sin b$.

(109.) From this, $\tan e . \cot E = \sin b'$, or $\cot c . \tan b = \cos A$; and $\cot c . \tan a = \cos B$.

(110.) These equations comprehend every case of right-angled spherical triangles; that is, if any two parts besides the right angle be given, any one of the remaining parts can be found by a short logarithmic calculation. In the opinion of Delambre (and no one was better qualified by experience to give an opinion) these theorems are best recollected by the practical calculator in their unconnected form. For common purposes, however, a technical memory has been invented, under the title of Naper's rules for Circular Parts, which we shall now describe.

(111.) The five circular parts are the two sides, the complement of the hypotenuse, and the complements of the angles. Any one of these is called a *middle part;* the two next it are then called the *adjacent parts*, and the two remaining ones the *opposite parts.* The two rules are then as follows· the sine of the middle part = product of tangents of adjacent parts; and the sine of the middle part = product of cosines of opposite parts.

(112.) These rules are proved to be true only by showing that they comprehend all the equations which we have just found. We shall leave to the reader the labour of examining every case.

(113.) It was observed in (100) that the polar triangle, corresponding to a right-angled triangle, is a quadrantal triangle. Naper's rules then may be applied to quadrantal triangles, if we take for the circular parts the complements of the sides, the complement of the angle opposite the quadrant, and the two angles. But as there is some difficulty in the determination of the signs, it will, perhaps, be found more convenient to make use of the general formulæ of (102) and (103), which for this case are always much simplified.

(114.) We shall now examine whether any of these solutions are ambiguous. And for this purpose, as before, we shall attend only to those whose values are given by the values of their sines. Now it is easily seen, that if A and a be given, B, b, and c are all given by their sines; and this case therefore is ambiguous, there being nothing which will enable us to determine whether the smallest corresponding arcs, or their supplements, are to be taken. In fact, the triangles A B C and A′ B C, (fig. 21), will equally satisfy the given conditions, since the angle at A′ = that at A.

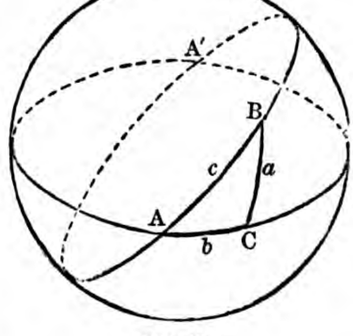

Fig. 21.

(115.) If A and c be given, a is given by its sine. Since, however, $\tan a = \sin b \,.\, \tan A$, and the tangent becomes negative when the arc is greater than 90°, and since $\sin b$ is always positive, (as b must be less than 180°), a must be greater or less than 90°, as A is greater or less than 90°, which removes the apparent ambiguity. If a and c be given to find A, the same remark applies.

(116.) We proceed to find formulæ of solution for all spherical triangles. Given the three sides to find the angles. We have seen

(103) that $$\cos C = \frac{\cos c - \cos a \,.\, \cos b}{\sin a \,.\, \sin b}.$$

This formula is not adapted to logarithmic calculation.

But $1 + \cos C$, or $2\cos^2\frac{C}{2} = \frac{\cos c - (\cos a \, . \cos b - \sin a \ \sin b)}{\sin a \, . \sin b}$

$$= \frac{\cos c - \cos(a+b)}{\sin a \, . \sin b} = \frac{2\sin\frac{a+b+c}{2} \, . \sin\frac{a+b-c}{2}}{\sin a \, . \sin b};$$

or putting $s = \frac{a+b+c}{2}$, $\cos^2\frac{C}{2} = \frac{\sin s \, . \sin(s-c)}{\sin a \, . \sin b}$.

Again, $1 - \cos C$, or $2\sin^2\frac{C}{2} = \frac{(\cos a \, . \cos b + \sin a \, . \sin b) - \cos c}{\sin a \, . \sin b}$

$$= \frac{\cos(a-b) - \cos c}{\sin a \, . \sin b} = \frac{2\sin\frac{b+c-a}{2} \, . \sin\frac{a+c-b}{2}}{\sin a \, . \sin b},$$

$$\sin^2\frac{C}{2} = \frac{\sin(s-a) \, . \sin(s-b)}{\sin a \, . \sin b}.$$

Dividing this by the former, $\tan^2\frac{C}{2} = \frac{\sin(s-a) \, . (s-b)}{\sin s \, . \sin(s-c)}$.

Multiplying them together, since $\sin C = 2\sin\frac{C}{2}\cos\frac{C}{2}$,

$$\sin^2 C = \frac{4 \, . \sin s \, . \sin(s-a) \, . \sin(s-b) \, . \sin(s-c)}{\sin^2 a \, . \sin^2 b}.$$

With all these forms logarithms can conveniently be used.

(117.) Given two sides (a, b) and the included angle (C) to find the other parts. From the expressions just found,

$$\tan\frac{A}{2} = \sqrt{\frac{\sin(s-b) \, . \sin(s-c)}{\sin s \, . \sin(s-a)}};$$

$$\tan\frac{B}{2} = \sqrt{\frac{\sin(s-a) \, . \sin(s-c)}{\sin s \, . \sin(s-b)}},$$

therefore $$\tan\frac{A+B}{2} = \frac{\tan\frac{A}{2} + \tan\frac{B}{2}}{1 - \tan\frac{A}{2} \, . \tan\frac{B}{2}}$$

$$=\sqrt{\frac{\sin(s-c)}{\sin s}}\cdot\frac{\sqrt{\dfrac{\sin(s-b)}{\sin(s-a)}}+\sqrt{\dfrac{\sin(s-a)}{\sin(s-b)}}}{1-\dfrac{\sin(s-c)}{\sin s}}$$

$$=\sqrt{\frac{\sin s\,.\,\sin(s-c)}{\sin(s-a)\,.\,\sin(s-b)}}\times\frac{\sin(s-b)+\sin(s-a)}{\sin s-\sin(s-c)}$$

$$=\cot\frac{C}{2}\cdot\frac{2\sin\dfrac{c}{2}\,.\,\cos\dfrac{a-b}{2}}{2\cos\dfrac{a+b}{2}\,.\,\sin\dfrac{c}{2}}=\frac{\cos\dfrac{a-b}{2}}{\cos\dfrac{a+b}{2}}\cot\frac{C}{2}.$$

Similarly, $$\tan\frac{A-B}{2}=\frac{\tan\dfrac{A}{2}-\tan\dfrac{B}{2}}{1+\tan\dfrac{A}{2}\tan\dfrac{B}{2}}$$

$$=\cot\frac{C}{2}\times\frac{\sin(s-b)-\sin(s-a)}{\sin s+\sin(s-c)}=\frac{\sin\dfrac{a-b}{2}}{\sin\dfrac{a+b}{2}}\cot\frac{C}{2}.$$

The sum and difference of A and B being thus found, A and B will be determined. The third side will be found by the proportion of (102).

(118.) It is, however, very frequently desirable to find the third side without finding the angles. Now,

$$\cos c\ (103)=\cos a\,.\,\cos b+\sin a\,.\,\sin b\,.\,\cos C,$$

or $$\text{versin}\ c=1-\cos c=1-\cos a\,.\,\cos b-\sin a\,.\,\sin b\,.\,\cos C$$

$$=1-(\cos a\,.\,\cos b+\sin a\,.\,\sin b)+\sin a\,.\,\sin b\,.\,\text{versin}\ C$$

$$=\text{versin}\,(a-b)+\sin a\,.\,\sin b\,.\,\text{versin}\ C$$

$$=\text{versin}\,(a-b)\,.\left(1+\frac{\sin a\,.\,\sin b\,.\,\text{versin}\ C}{\text{versin}\,(a-b)}\right).$$

Make $\frac{\sin a \,.\, \sin b \,.\, \text{versin}\, C}{\text{versin}\,(a-b)} = \tan^2\theta$;

then $$\text{versin}\, c = \text{versin}\,(a-b) \,.\, \sec^2\theta.$$

Or thus, $$\cos C = 2\cos^2\frac{C}{2} - 1;$$

therefore $\cos c = \cos a \,.\, \cos b - \sin a \,.\, \sin b + 2\sin a \,.\, \sin b \,.\, \cos^2\frac{C}{2}$

$$= \cos(a+b) + 2\sin a \,.\, \sin b \,.\, \cos^2\frac{C}{2};$$

therefore $1 - \cos c$, or $2\sin^2\frac{c}{2} = 1 - \cos(a+b) - 2\sin a \,.\, \sin b \,.\, \cos^2\frac{C}{2}$,

and $$\sin^2\frac{c}{2} = \sin^2\frac{a+b}{2} - \sin a \,.\, \sin b \,.\, \cos^2\frac{C}{2}.$$

Let $\sin a \,.\, \sin b \,.\, \cos^2\frac{C}{2} = \sin^2\theta$; then

$$\sin^2\frac{c}{2} = \sin^2\frac{a+b}{2} - \sin^2\theta =, \text{by (39)}, \sin\left(\frac{a+b}{2}+\theta\right) . \sin\left(\frac{a+b}{2}-\theta\right).$$

(119.) The following theorem is frequently useful. We have found $\cos A = \frac{\cos a - \cos b \,.\, \cos c}{\sin b \,.\, \sin c}$;

also $$\cos c = \cos a \,.\, \cos b + \sin a \,.\, \sin b \,.\, \cos C;$$

substituting this in the numerator,

$$\cos A = \frac{\cos a - \cos^2 b \,.\, \cos a - \sin b \,.\, \cos b \,.\, \sin a \,.\, \cos C}{\sin b \,.\, \sin c}$$

$$= \frac{\cos a \,.\, \sin b - \cos b \,.\, \sin a \,.\, \cos C}{\sin c};$$

and $\sin c = \frac{\sin C \,.\, \sin a}{\sin A}$,

therefore $\cos A = \sin A \,.\, \frac{\cos a \,.\, \sin b - \cos b \,.\, \sin a \,.\, \cos C}{\sin C \,.\, \sin a}$,

or $$\cot A \,.\, \sin C = \cot a \,.\, \sin b - \cos C \,.\, \cos b.$$

This formula is chiefly useful for finding the corresponding small variations of the parts of a spherical triangle. It may also be used to determine A:

thus, $$\cot A = \frac{\cot a}{\sin C} \times \left(\sin b - \frac{\cos C}{\cot a}\cos b\right);$$

let $\frac{\cos C}{\cot a} = \tan \theta$, then

$$\cot A = \frac{\cot a}{\sin C \cdot \cos \theta} (\sin b \cos \theta - \cos b \sin \theta) = \frac{\cot a \cdot \sin (b - \theta)}{\sin C \cdot \cos \theta}.$$

(120.) Suppose two angles and the included side (A, B, c) given, To find the remaining parts. Take the polar triangle; let a', b', c', be the sides of which the points A, B, C, are the poles; A', B', C', the opposite angles. Then, (97), $c' = \pi - C$, $C' = \pi - c$.

Then $$\tan \frac{A' + B'}{2} = \cot \frac{C'}{2} \cdot \frac{\cos \frac{a' - b'}{2}}{\cos \frac{a' + b'}{2}}, \text{ (117)},$$

or $$-\tan \frac{a+b}{2} = \tan \frac{c}{2} \cdot \frac{\cos \frac{B - A}{2}}{-\cos \frac{A + B}{2}},$$

or $$\tan \frac{a+b}{2} = \tan \frac{c}{2} \cdot \frac{\cos \frac{A - B}{2}}{\cos \frac{A + B}{2}}.$$

Similarly, $$\tan \frac{A' - B'}{2} = \cot \frac{C'}{2} \cdot \frac{\sin \frac{a' - b'}{2}}{\sin \frac{a' + b'}{2}},$$

or $$\tan \frac{a-b}{2} = \tan \frac{c}{2} \cdot \frac{\sin \frac{A - B}{2}}{\sin \frac{A + B}{2}}.$$

The sides being thus found, the third angle may be found by the proportion of (102). If it be wished to have the third angle independently, the formulæ of (118) may be adapted in the same way.

(121.) If we divide one of the equations in (117) or (120) by the other, we find

$$\frac{\tan \frac{a+b}{2}}{\tan \frac{a-b}{2}} = \frac{\tan \frac{A+B}{2}}{\tan \frac{A-B}{2}}.$$

(122.) If two sides be given and an angle adjacent to one, then another angle is found by (102), and the third side by (120), or the third angle by (117). In this case the solution is ambiguous, under the same circumstances as in the corresponding case of plane triangles. If two angles and an adjacent side, B, C, *b*, (fig. 22), be given, the process is the same. In this case, when C is greater than B, either of the triangles C A B, C A B′ (in which B′ A produced makes A D = A B) satisfies the given conditions. These are the only ambiguous cases of oblique-angled spherical triangles.

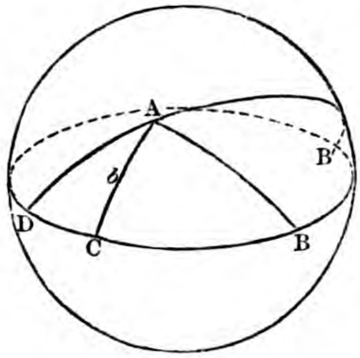

Fig. 22.

(123.) If the three angles be given, the formulæ of (116) may be applied to the polar triangle, and the sides of the given triangle may be found. This, however, is a case which never occurs in any applications of trigonometry.

Numerous examples of the solution of spherical triangles will be found in works on astronomy and the other subjects to which it is practically applied, but their introduction here would render necessary a mass of explanation and detail foreign to the scheme of the work.

SECTION VII.

ON SMALL CORRESPONDING VARIATIONS OF THE PARTS OF TRIANGLES.

(124.) It is frequently desirable to ascertain the effect which will be produced on one part of a triangle by the variation of another, all the rest remaining unvaried. To estimate the probable effect of error in observation; to reduce observations made in one situation to what they would be in a situation little distant; to take account of refraction, parallax, &c., this theory is absolutely necessary. We shall, therefore, give the general method of finding these corresponding variations.

(125.) In almost all cases expressions may be conveniently found by writing down two equations, one of which results from giving to the quantities contained in the other the variations which they are supposed to undergo, and then taking their difference. And this method has the advantage of showing precisely the magnitude of the error made by any further simplification. It will be best illustrated by examples.

(126.) The height of a building is found by measuring a horizontal line from its base, and at the extremity observing the apparent altitude; and the angle is liable to a small error of observation. In this case, if a be the measured distance, θ the angle, x the height, we have

$$x = a \,.\, \tan \theta.$$

And if giving to θ the variation, $\delta\theta$ would produce in x the variation δx, we have

$$x + \delta x = a \,.\, \tan(\theta + \delta\theta).$$

Subtracting the former equation,

$$\delta x = a \left\{ \tan(\theta + \delta\theta) - \tan\theta \right\} =, \text{ by (42)}, \; a \frac{\sin \delta\theta}{\cos\theta \,.\, \cos(\theta + \delta\theta)}.$$

Now, if we suppose $\delta\theta$ to be very small, we may put $\delta\theta$ instead of $\sin\delta\theta$, and $\cos\theta$ instead of $\cos(\theta + \delta\theta)$, without sensible error;

then

$$\delta x = \frac{a\,\delta\theta}{\cos^2 . \theta}.$$

Here $\delta\theta$ is supposed to be expressed by the length of the corresponding arc to radius 1. If it $= n$ seconds, then for $\delta\theta$ we must put

$n \times 0{,}000004848$, (4), and $\delta x = \dfrac{a \cdot n \cdot 0{,}000004848}{\cos^2\theta}$ very nearly.

(127.) If it were wished to determine a, so that the error should be a *minimum*, it must be observed that a though determinate is not constant, but $= x \cdot \cot\theta$, whence

$$\delta x = \frac{x\,\delta\theta}{\sin\theta \cdot \cos\theta} = \frac{2\,x\,\delta\theta}{\sin 2\theta},$$

which is least when $\sin 2\theta$ is greatest, or $2\theta = \frac{\pi}{2}$, or $\theta = \frac{\pi}{4}$.

(128.) Suppose in a right-angled spherical triangle, C being the right angle, A is given, To find the variation of a when c receives a small variation. Here (106) $\sin a = \sin A \cdot \sin c$; hence

$$\sin(a + \delta a) = \sin A \cdot \sin(c + \delta c);$$

taking the difference,

$$\sin(a + \delta a) - \sin a = \sin A \left\{\sin(c + \delta c) - \sin c\right\},$$

or $$2\cos\left(a + \frac{\delta a}{2}\right) \cdot \sin\frac{\delta a}{2} = 2\sin A \cdot \cos\left(c + \frac{\delta c}{2}\right) \cdot \sin\frac{\delta c}{2},$$

and $$\sin\frac{\delta a}{2} = \frac{\sin A \cdot \cos\left(c + \frac{\delta c}{2}\right)}{\cos\left(a + \frac{\delta a}{2}\right)} \sin\frac{\delta c}{2},$$

or, if δa and δc be very small,

$$\delta a = \frac{\sin A \cdot \cos c}{\cos a}\,\delta c = \sin A \cdot \cos b \cdot \delta c, \text{ or } = \frac{\tan a}{\tan c}\,\delta c.$$

If m be the number of seconds in δa, n that in δc, $m = \dfrac{\tan a}{\tan c}\,n$.

(129.) The consideration of particular cases of this last problem shows that we must be cautious in applying to any extent the simplifications which were there introduced from considering δc as small. Suppose $c = 90°$; it would seem that $\delta a = 0$. Taking, however, the original expression

$$\sin\frac{\delta a}{2} = \frac{\sin A \cdot \cos\left(c + \frac{\delta c}{2}\right)}{\cos\left(a + \frac{\delta a}{2}\right)} \cdot \sin\frac{\delta c}{2},$$

we may observe that, when $c = \frac{\pi}{2}$,

$$\cos\left(c + \frac{\delta c}{2}\right) = -\sin\frac{\delta c}{2}, \text{ by (23) and (34);}$$

therefore $$\sin\frac{\delta a}{2} = \frac{-\sin A}{\cos\left(a + \frac{\delta a}{2}\right)}\sin^2\frac{\delta c}{2}.$$

Making $\frac{\delta c}{2}$ very small, $\frac{\delta a}{2} = -\frac{\sin A}{\cos a} \cdot \frac{\overline{\delta c}|^2}{4}$,

or $$\delta a = -\frac{\sin A}{2\cos a}\overline{\delta c}|^2.$$

Here then $$m = -\frac{\sin A \times 0{,}000004848}{2\cos a}n^2$$

$$= -\tan A \times 0{,}000002424 \times n^2, \text{ since } a \text{ now} = A.$$

(130.) Given two sides (a, b) of a spherical triangle, and the included angle (C) to find the variation produced in c by the variation of C. Here

$$\cos c = \cos a \,.\, \cos b + \sin a \,.\, \sin b \,.\, \cos C, \text{ (103)},$$

and $\cos(c + \delta c) = \cos a \,.\, \cos b + \sin a \,.\, \sin b \,.\, \cos(C + \delta C)$.

Subtracting the latter,

$$2\sin\left(c + \frac{\delta c}{2}\right) . \sin\frac{\delta c}{2} = 2\sin a \,.\, \sin b \,.\, \sin\left(C + \frac{\delta C}{2}\right) . \sin\frac{\delta C}{2}.$$

If δC be small, and if C or c be not small,

then $$\sin c \,.\, \delta c = \sin a \,.\, \sin b \,.\, \sin C \,.\, \delta C \text{ nearly},$$

or $$\delta c = \frac{\sin a \,.\, \sin b \,.\, \sin C}{\sin c} \times \delta C = \sin B \,.\, \sin a \,.\, \delta C.$$

If m and n be the number of seconds in δc and δC,

$$m = \sin B \,.\, \sin a \,.\, n.$$

If $C = 0$, then

$$2\sin\left(c + \frac{\delta c}{2}\right) . \sin\frac{\delta c}{2} = 2\sin a \,.\, \sin b \,.\, \sin^2\frac{\delta C}{2};$$

and supposing δC very small,

$$\sin c \,.\, \frac{\delta c}{2} = \sin a \,.\, \sin b \,.\, \frac{\overline{\delta C}|^2}{4},$$

or $$\delta c = \frac{\sin a \,.\, \sin b}{2\sin c}\overline{\delta C}|^2,$$

or $$m = \frac{\sin a \,.\, \sin b \times 0{,}000004848}{2\sin c} \times n^2.$$

(131.) With the same data, to find the variation in A.

Here (119) $\cot A . \sin C = \cot a . \sin b - \cos C . \cos b$,

and
$$\cot (A + \delta A) . \sin (C + \delta C)$$
$$= \cot a . \sin b - \cos (C + \delta C) . \cos b;$$

subtracting the former, $\cot (A + \delta A) . \sin (C + \delta C) - \cot A . \sin C$
$$= \cos b . \left\{\cos C - \cos (C + \delta C)\right\}.$$

Now,
$$\cot (A + \delta A) . \left\{\sin (C + \delta C) - \sin C\right\}$$
$$= \cot (A + \delta A) . 2 \cos \left(C + \frac{\delta C}{2}\right) . \sin \frac{\delta C}{2};$$

also $\sin C \left\{\cot (A + \delta A) - \cot A\right\} = - \sin C \dfrac{\sin . \delta A}{\sin A . \sin (A + \delta A)}$;

adding these together, $\cot (A + \delta A) . (\sin C + \delta C) - \cot A . \sin C$
$$= 2 \cot (A + \delta A) . \cos \left(C + \frac{\delta C}{2}\right) . \sin \frac{\delta C}{2} - \sin C . \frac{\sin \delta A}{\sin A . \sin (A + \delta A)}.$$

And $\cos C - \cos (C + \delta C) = 2 \sin \left(C + \dfrac{\delta C}{2}\right) . \sin \dfrac{\delta C}{2}$;

substituting in the equation, and supposing δC and δA very small,
$$\cot A . \cos C . \delta C - \frac{\sin C}{\sin^2 A} \delta A = \cos b . \sin C . \delta C,$$

and
$$\delta A = \frac{\sin^2 A}{\sin C} (\cot A \cos C - \cos b . \sin C) . \delta C;$$

or, if p be the number of seconds in δA,
$$p = \frac{\sin^2 A}{\sin C} (\cot A \cos C - \cos b . \sin C) \times n.$$

Putting for cot A its value, this is easily changed into
$$p = - \frac{\sin^2 A}{\sin C} . \cot B . \frac{\sin b}{\sin a} n = - \frac{\sin A . \cos B}{\sin C} n.$$

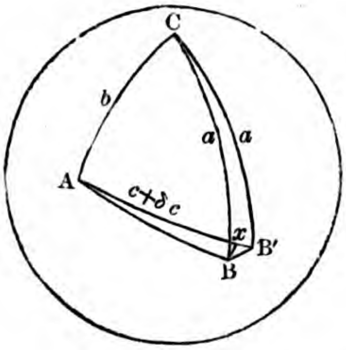

Fig. 23.

(132.) The principle and the mode of its application is now sufficiently evident. We must, however, remark that in many cases the corresponding variations may be easily found by geometrical considerations. Thus, for the problem of (130), let A B C (fig. 23) be the triangle, and by the variation of C let it be changed to A B′ C; if B x be supposed to be drawn perpendicular to A B′, then A x will ultimately = A B, and therefore x B′ = δc. Now δc = B B′ sin B′ B x

$= \mathrm{B\,B'} \sin \mathrm{C\,B\,A}$ (since $\mathrm{C\,B\,B'}$ is a right angle, as C is the pole of $\mathrm{B\,B'}$ (86), and therefore $\mathrm{C\,B\,B'} = x\,\mathrm{B\,A}$); but

$$\mathrm{B\,B'} = \sin a \,.\, \mathrm{B\,C\,B'} \text{ by (90)} = \sin a \,.\, \delta \mathrm{C},$$

therefore $\delta c = \sin a \,.\, \sin \mathrm{C\,B\,A} \,.\, \delta \mathrm{C} = \sin \mathrm{B} \,.\, \sin a \,.\, \delta \mathrm{C}$, as in (130).

And if the variation of A were required, we should have

$$\delta \mathrm{A} = \frac{\mathrm{B}\,x}{\sin c} = \frac{\mathrm{B\,B'} \,.\, \cos \mathrm{B'\,B}\,x}{\sin c} = \frac{\sin a \,.\, \cos \mathrm{B} \,.\, \delta \mathrm{C}}{\sin c},$$

or
$$= \frac{\sin \mathrm{A} \,.\, \cos \mathrm{B}}{\sin \mathrm{C}} \delta \mathrm{C}$$

for the quantity by which A is diminished, as in (131).

(133.) The geometrical method then can be applied with great ease to those examples in which the variation of one element is expressed in terms of the first power of the variation of another element, but it can very seldom be applied to those cases in which, as in (129), the variation of one depends on the square of the variation of the other. Another method will hereafter be described, not however preferable in general to the first given here.

SECTION VIII.

INVESTIGATIONS REQUIRING A HIGHER ANALYSIS THAN THE PRECEDING.

(134.) The preceding sections have referred to nothing more difficult than the most common propositions of plane geometry and algebra, and one or two theorems of solid geometry. In this section it is proposed to comprehend some of those expressions which require for their demonstration some of the higher parts of analysis, particularly the differential calculus, and the calculus of finite differences.

(135.) To express generally $\cos n x$ in a series proceeding by powers of $\cos x$. If we observe the manner in which the expressions of (38) are successively formed, we shall easily see that $\cos n x$, n being a positive integer, will always be expressed by this form,

$$2^{n-1} \cos^n x + a \cos^{n-2} x + b \cos^{n-4} x + \&c.;$$

a, b, &c., being functions of n.

Also there will be no second term till $n=2$; no third term till $n=4$,&c.

Let $2 \cos n x = u_n$; $2 \cos x = p$; then $u_{n+1} = p \,.\, u_n - u_{n-1}$.

Assume then $u_n = p^n + A_n \,.\, p^{n-2} + B_n \,.\, p^{n-4} + C_n \,.\, p^{n-6} + \&c.$,

A_n, B_n, &c., being functions of n to be determined; then

$$u_{n+1} = p^{n+1} + A_{n+1} \,.\, p^{n-1} + B_{n+1} \,.\, p^{n-3} + C_{n+1} \,.\, p^{n-5} + \&c.;$$

$$u_{n-1} = p^{n-1} + A_{n-1} \,.\, p^{n-3} + B_{n-1} \,.\, p^{n-5} + C_{n-1} \,.\, p^{n-7} + \&c.$$

Substituting these expressions in the equation above, and equating the coefficients of similar powers, we have these equations,

$$A_{n+1} = A_n - 1;\; B_{n+1} = B_n - A_{n-1};\; C_{n+1} = C_n - B_{n-1},\; \&c.;$$

or, since $A_{n+1} - A_n = \Delta \,.\, A_n$, &c.,

$$\Delta \,.\, A_n = -1;\; \Delta \,.\, B_n = - A_{n-1};\; \Delta\, C_n = - B_{n-1},\; \&c.$$

Integrating the first, we have $A_n = -n + C$;

and since $2 \cos 2 x = (2 \cos x)^2 - 2$, we must have $A_2 = -2$, therefore $C = 0$, and $A_n = -n$.

(It will be remarked, that we have not found the correction by making $n = 1$, because the equation $A_2 = A_1 - 1$ is not true, the value of u_0 or $2 \cos 0$ being not 1 but 2. After this, however, the equation $A_{n+1} = A_n - 1$ always holds; A_2 therefore is the first quantity to which the general value of A_n can be applied. The other equations $B_{n+1} = B_n - A_{n-1}$, &c., are true without any exception.)

Hence $A_{n-1} = -(n-1)$, therefore $\Delta B_n = (n-1) = (n-2) + 1$;

integrating, $$B_n = \frac{(n-2) \cdot (n-3)}{1 \cdot 2} + n + C;$$

making this $= 0$ when $n = 3$,

$$B_n = \frac{(n-2) \cdot (n-3)}{1 \cdot 2} + (n-3) = \frac{n \cdot (n-3)}{1 \cdot 2}.$$

Hence $$-B_{n-1} = -\frac{(n-1) \cdot (n-4)}{1 \cdot 2}$$

$$= -\frac{(n-3) \cdot (n-4)}{1 \cdot 2} - 2\frac{n-4}{2} = \Delta C_n;$$

therefore $$C_n = -\frac{(n-3) \cdot (n-4) \cdot (n-5)}{1 \cdot 2 \cdot 3} - \frac{(n-4) \cdot (n-5)}{1 \cdot 2}$$

$$= -\frac{n \cdot (n-4) \cdot (n-5)}{1 \cdot 2 \cdot 3},$$

which needs no correction, as it vanishes when $n = 5$.

Continuing the process, we find

$$D_n = \frac{n \cdot (n-5) \cdot (n-6) \cdot (n-7)}{1 \cdot 2 \cdot 3 \cdot 4}, \text{ \&c.};$$

hence $$u_n = p^n - n \cdot p^{n-2} + \frac{n \cdot (n-3)}{1 \cdot 2} p^{n-4} - \text{\&c.};$$

or $$2 \cos nx = (2 \cos x)^n - n\,(2 \cos x)^{n-2} + \frac{n \cdot (n-3)}{1 \cdot 2} (2 \cos x)^{n-4}$$

$$- \frac{n \cdot (n-4) \cdot (n-5)}{1 \cdot 2 \cdot 3} (2 \cos x)^{n-6}$$

$$+ \frac{n \cdot (n-5) \cdot (n-6) \cdot (n-7)}{1 \cdot 2 \cdot 3 \cdot 4} (2 \cos x)^{n-8} - \text{\&c.}$$

Of this important theorem we believe this is the simplest demonstration that has yet been given.

(136.) If n be even and $= 2m$,

the last term will be $\pm \dfrac{2m \,.\, (m-1) \,.\, (m-2) \ldots . 1}{1 \,.\, 2 \,.\, 3 \ldots . m} = \pm 2$;

$$\text{the last but one} = \mp \frac{2m \,.\, m \,.\, (m-1) \ldots . 4 \,.\, 3}{1 \,.\, 2 \,.\, 3 \ldots . (m-1)} (2 \cos x)^2$$
$$= \mp m^2 \,.\, (2 \cos x)^2;$$

$$\text{the last but two} = \pm \frac{2m \,.\, (m+1) \,.\, m \,.\, (m-1) \,.\,.\, 6 \,.\, 5}{1 \,.\, 2 \,.\, 3 \ldots . (m-2)} (2 \cos x)^4$$
$$= \pm \frac{2m^2 \,.\, (m^2-1)}{1 \,.\, 2 \,.\, 3 \,.\, 4} (2 \cos x)^4, \text{ \&c.}$$

$$\text{Hence } \cos nx = \pm \left\{1 - \frac{n^2}{1 \,.\, 2} \cos^2 x + \frac{n^2 \,.\, (n^2-4)}{1 \,.\, 2 \,.\, 3 \,.\, 4} \cos^4 x \right.$$
$$\left. - \frac{n^2 \,.\, (n^2-4) \,.\, (n^2-16)}{1 \,.\, 2 \,.\, 3 \,.\, 4 \,.\, 5 \,.\, 6} \cos^6 x + \text{\&c.}\right\}$$

the upper sign to be taken when m is even, or n divisible by 4.
If n be odd or of the form $(2m+1)$, the last term will be

$$\pm \frac{(2m+1) \,.\, m \,.\, (m-1) \ldots . 2}{1 \,.\, 2 \,.\, 3 \ldots . m} 2 \cos x$$
$$= \pm (2m+1) \,.\, 2 \cos x = \pm n \,.\, 2 \cos x;$$

$$\text{the last but one} = \mp \frac{(2m+1) \,.\, (m+1) \,.\, m \,.\, (m-1) \ldots . 4}{1 \,.\, 2 \,.\, 3 \ldots . (m-1)} (2 \cos x)^3$$
$$= \mp \frac{(2m+1) \,.\, (m+1) \,.\, m}{1 \,.\, 2 \,.\, 3} (2 \cos x)^3$$
$$= \mp \frac{n \,.\, (n^2-1)}{1 \,.\, 2 \,.\, 3} 2 \cos^3 x, \text{ \&c.}$$

Hence, when n is odd, $\cos nx =$

$$\mp \left\{ n \cos x - \frac{n \,.\, (n^2-1)}{1 \,.\, 2 \,.\, 3} \cos^3 x + \frac{n \,.\, (n^2-1) \,.\, (n^2-9)}{1 \,.\, 2 \,.\, 3 \,.\, 4 \,.\, 5} \cos^5 x - \text{\&c.} \right\},$$

the upper sign being taken when n is of the form $4s+1$, and the lower when of the form $4s+3$.

(137.) Let $x = \dfrac{\pi}{2} - y$; then $\cos x = \sin y$, and $\cos nx$

$$= \cos\left(\frac{n\pi}{2} - ny\right) = \cos \frac{n\pi}{2} \cos ny + \sin \frac{n\pi}{2} \sin ny.$$

If n be divisible by 4, this $= \cos ny$; if only divisible by 2, it $= -\cos ny$. Hence in all cases, n being even,

$$\cos n y = 1 - \frac{n^2}{1.2} \cdot \sin^2 y + \frac{n^2(n^2-4)}{1.2.3.4} \sin^4 y - \&c.$$

If n be of the form $4s+1$, $\cos nx = \sin ny$; if of the form $4s+3$ $\cos nx = -\sin ny$. Hence in all cases, n being odd,

$$\sin n y = n \sin y - \frac{n(n^2-1)}{1.2.3} \sin^3 y + \&c.$$

(**138.**) Differentiating the first equation of the last article we find, n being even,

$$\sin n y = \cos y \left\{ n \sin y - \frac{n(n^2-4)}{1.2.3} \sin^3 y \right.$$
$$\left. + \frac{n(n^2-4)(n^2-16)}{1.2.3.4.5} \sin^5 y - \&c. \right\}$$

By similar operations we may from these deduce other formulæ.

(**139.**) Let $ny = z$; then (n even)

$$\cos z = 1 - \frac{z^2}{1.2} \cdot \frac{\sin^2 y}{y^2} + \frac{z^4}{1.2.3.4} \cdot \frac{\left(1-\frac{4}{n^2}\right)\sin^4 y}{y^4}$$
$$- \frac{z^6}{1.2.3.4.5.6} \cdot \frac{\left(1-\frac{4}{n^2}\right) \cdot \left(1-\frac{16}{n^2}\right)\sin^6 y}{y^6} + \&c.$$

Suppose now n to be increased without limit; the expressions $1-\frac{4}{n^2}$, $1-\frac{16}{n^2}$, &c. approach to **1** as their limit; the fraction $\frac{\sin y}{y}$ also has 1 for its limit.

Hence $\cos z = 1 - \dfrac{z^2}{1.2} + \dfrac{z^4}{1.2.3.4} - \dfrac{z^6}{1.2.3.4.5.6} + \&c.$

(**140.**) Again, (n odd)

$$\sin z = z \cdot \frac{\sin y}{y} - \frac{z^3}{1.2.3} \cdot \frac{\left(1-\frac{1}{n^2}\right)\sin^3 y}{y^3}$$
$$+ \frac{z^5}{1.2.3.4.5} \cdot \frac{\left(1-\frac{1}{n^2}\right)\left(1-\frac{9}{n^2}\right)\sin^5 y}{y^5} - \&c.$$

Increasing n without limit,

$$\sin z = z - \frac{z^3}{1.2.3} + \frac{z^5}{1.2.3.4.5} - \&c.$$

(**141.**) Now we may remark, that if we expand $e^{x\sqrt{-1}}$ and $e^{-x\sqrt{-1}}$ (e being the base of Naperian logarithms $= 2{,}7182818$) in the same way in which we expand e^x, we have

$$e^{x\sqrt{-1}} = 1 + \frac{x\sqrt{-1}}{1} - \frac{x^2}{1.2} - \frac{x^3\sqrt{-1}}{1.2.3} + \frac{x^4}{1.2.3.4} + \&c.$$

$$e^{-x\sqrt{-1}} = 1 - \frac{x\sqrt{-1}}{1} - \frac{x^2}{1.2} + \frac{x^3\sqrt{-1}}{1.2.3} + \frac{x^4}{1.2.3.4} - \&c.$$

Adding them,

$$e^{x\sqrt{-1}} + e^{-x\sqrt{-1}} = 2\left(1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \&c.\right) = 2\cos x,$$

or
$$\cos x = \frac{e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}}{2}.$$

Subtracting,

$$e^{x\sqrt{-1}} - e^{-x\sqrt{-1}} = 2\sqrt{-1}\left\{x - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} - \&c.\right\}$$
$$= 2\sqrt{-1}\,.\sin x,$$

or
$$\sin x = \frac{e^{x\sqrt{-1}} - e^{-x\sqrt{-}}}{2\sqrt{-1}}.$$

These expressions are to be regarded as having no other meaning than this; if expanded according to the rules by which we expand possible algebraic quantities, they would produce the series for $\cos x$ and $\sin x$.

(**142.**) From these equations we have

$$e^{x\sqrt{-1}} = \cos x + \sqrt{-1}\,.\sin x;\ e^{-x\sqrt{-1}} = \cos x - \sqrt{-1}\,.\sin x.$$

Similarly, $\quad e^{y\sqrt{-1}} = \cos y + \sqrt{-1}\,.\sin y;$

multiplying this by $e^{x\sqrt{-1}}$,

$$e^{(x+y)\sqrt{-1}} = (\cos x + \sqrt{-1}\sin x)\ (\cos y + \sqrt{-1}\,.\sin y).$$

But $\quad e^{(x+y)\sqrt{-1}} = \cos(x+y) + \sqrt{-1}\,.\sin(x+y);$

hence we have this very remarkable formula,

$$(\cos x + \sqrt{-1}\,.\sin x)\,.\,(\cos y + \sqrt{-1}\,.\sin y)$$
$$= \cos(x+y) + \sqrt{-1}\,.\sin(x+y).$$

The same result will be obtained by actually multiplying together the two factors.

If we suppose y successively $= x$, $2x$, &c., we shall have

$$(\cos x + \sqrt{-1} \,.\, \sin x)^n = \cos nx + \sqrt{-1} \,.\, \sin nx,$$

which implies that n is an integer.

Or thus, $e^{nx\sqrt{-1}} = (e^{x\sqrt{-1}})^n$, that is,

$$\cos nx + \sqrt{-1} \,.\, \sin nx = (\cos x + \sqrt{-1} \,.\, \sin x)^n,$$

whether n be whole or fractional.

Similarly, $\cos nx - \sqrt{-1} \,.\, \sin nx = (\cos x - \sqrt{-1} \,.\, \sin x)^n$.

This theorem is due to Demoivre.

(143.) Expanding the two last expressions, adding them together, and dividing by two, we have

$$\cos nx = \cos^n x - \frac{n \,.\, (n-1)}{1 \,.\, 2} \cos^{n-2} x \,.\, \sin^2 x$$

$$+ \frac{n \,.\, (n-1) \,.\, (n-2) \,.\, (n-3)}{1 \,.\, 2 \,.\, 3 \,.\, 4} \,.\, \cos^{n-4} x \,.\, \sin^4 x - \text{\&c.}$$

$$= \cos^n x \left\{ 1 - \frac{n \,.\, (n-1)}{1 \,.\, 2} \tan^2 x \right.$$

$$\left. + \frac{n \,.\, (n-1) \,.\, (n-2) \,.\, (n-3)}{1 \,.\, 2 \,.\, 3 \,.\, 4} \tan^4 x - \text{\&c.} \right\}$$

Subtracting and dividing by $2\sqrt{-1}$,

$$\sin nx = n \,.\, \cos^{n-1} x \,.\, \sin x - \frac{n \,.\, (n-1) \,.\, (n-2)}{1 \,.\, 2 \,.\, 3} \cos^{n-3} x \,.\, \sin^3 x + \text{\&c.}$$

$$= \cos^n x \left\{ n \tan x - \frac{n \,.\, (n-1) \,.\, (n-2)}{1 \,.\, 2 \,.\, 3} \tan^3 x + \text{\&c.} \right\}$$

Dividing the latter by the former, $\tan nx$

$$= \frac{n \tan x - \dfrac{n \,.\, (n-1) \,.\, (n-2)}{1 \,.\, 2 \,.\, 3} \tan^3 x + \text{\&c.}}{1 - \dfrac{n \,.\, (n-1)}{1 \,.\, 2} \tan^2 x + \dfrac{n \,.\, (n-1) \,.\, (n-2) \,.\, (n-3)}{1 \,.\, 2 \,.\, 3 \,.\, 4} \tan^4 x - \text{\&c.}}.$$

(144.) In (142) suppose such a value to be given to nx that $\sin nx = 0$, $\cos nx = 1$; then $nx = 0$, or 2π, or 4π, &c., and $x = 0$, or $\frac{2\pi}{n}$, or $\frac{4\pi}{n}$, &c. Hence the following equations are true,

$$\overline{1}|^n = 1;\ \left(\cos \frac{2\pi}{n} \pm \sqrt{-1} \sin \frac{2\pi}{n}\right)^n = 1;$$

$$\left(\cos \frac{4\pi}{n} \pm \sqrt{-1} \,.\, \sin \frac{4\pi}{n}\right)^n = 1;\ \text{\&c.}$$

The quantities within the brackets are therefore roots of the equation $z^n = 1$, or $z^n - 1 = 0$. Hence we have for simple divisors of that equation,

$$z - 1,\ z - \cos\frac{2\pi}{n} - \sqrt{-1}\,.\sin\frac{2\pi}{n},$$

$$z - \cos\frac{2\pi}{n} + \sqrt{-1}\,.\sin\frac{2\pi}{n},\ \&c.;$$

or grouping in pairs the corresponding factors, the factors of

$z^n - 1$ are $z - 1,\ z^2 - 2z\,.\cos\frac{2\pi}{n} + 1,\ z^2 - 2z\,.\cos\frac{4\pi}{n} + 1$, &c.,

to be continued till the number of dimensions $= n$. If n be even, the last factor will be $z + 1$. In a similar way, we have for the factors of

$$z^n + 1,\ z^2 - 2z\cos\frac{\pi}{n} + 1,\ z^2 - 2z\cos\frac{3\pi}{n} + 1,\ \&c.,$$

to n dimensions. If n be odd, the last factor will be $z + 1$.

(**145**.) If we put $\frac{w}{a}$ for z, we have

$$w^n - a^n = (w - a)\,.\left(w^2 - 2wa\cos\frac{2\pi}{n} + a^2\right).$$

$$\left(w^2 - 2wa\cos\frac{4\pi}{n} + a^2\right),\ \&c.$$

to n dimensions, the last factor being $w + a$ if n be even. And

$$w^n + a^n = \left(w^2 - 2wa\cos\frac{\pi}{n} + a^2\right).\left(w^2 - 2wa\cos\frac{3\pi}{n} + a^2\right),$$

&c. to n dimensions, the last factor being $w + a$ if n be odd.

This is called, from the inventor, Cotes's theorem.

(**146**.) It is required to express $(\cos x)^n$ by the cosines of multiples of x. Here, n being an integer,

$$(\cos x)^n = \frac{1}{2^n}\left(e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}\right)^n$$

$$= \frac{1}{2^n}\left\{e^{nx\sqrt{-1}} + n\,.\,e^{(n-2)x\sqrt{-1}} + \frac{n\,.(n-1)}{1\,.\,2}e^{(n-4)x\sqrt{-1}} + \&c.\right.$$

$$\left. + \frac{n\,.(n-1)}{1\,.\,2}e^{-(n-4)x\sqrt{-1}} + n\,.\,e^{-(n-2)x\sqrt{-1}} + e^{-nx\sqrt{-1}}\right\}$$

$$= \frac{1}{2^n} \cdot \left\{ e^{nx\sqrt{-1}} + e^{-nx\sqrt{-1}} + n\left(e^{(n-2)x\sqrt{-1}} + e^{-(n-2)x\sqrt{-1}}\right) \right.$$

$$\left. + \frac{n.(n-1)}{1.2}\left(e^{(n-4)x\sqrt{-1}} + e^{-(n-4)x\sqrt{-1}}\right) + \&c. \right\}$$

$$= \frac{1}{2^{n-1}} \left\{ \cos nx + n\cos(n-2)x + \frac{n.(n-1)}{2}\cos(n-4)x + \&c. \right\}$$

The coefficients are the same as those in the first half of the expansion of $(a+b)^n$; but if n be even, the last term, which does not multiply a cosine, is half of the middle term in the expansion of the binomial.

(147.) As this formula is demonstrated entirely by means of imaginary symbols, we shall endeavour to explain how it happens that operations conducted by imaginary expressions can give correctly a real result. We know that

$$\left(\frac{e^y + e^{-y}}{2}\right)^n = \frac{1}{2^n} \cdot \{e^{ny} + e^{-ny} + n\left(e^{(n-2)y} + e^{-(n-2)y}\right) + \&c.\},$$

or

$$\left(1 + \frac{y^2}{1.2} + \frac{y^4}{1\ 2.3.4} + \&c.\right)^n$$

$$= \frac{1}{2^{n-1}} \cdot \left\{1 + \frac{n^2 y^2}{1\ 2} + \frac{n^4 y^4}{1.2.3.4} + \&c \right.$$

$$\left. + n\left(1 + \frac{(n-2)^2 . y^2}{1.2} + \frac{(n-2)^4 . y^4}{1.2.3.4} + \&c.\right) + \&c. \right\}$$

Now this is true for all values of y; and, consequently, if both sides were expanded, the expanded expressions would be *identically* equal; and therefore there would be still an equality if instead of y^2 we put $-x^2$, and operated upon it by the rules of common algebra. This would give us

$$\left(1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \&c.\right)^n = \frac{1}{2^{n-1}} \left\{1 - \frac{n^2 x^2}{1.2} + \frac{n^4 x^4}{1.2.3.4} - \&c. \right.$$

$$\left. + n\left(1 - \frac{(n-2)^2 x^2}{1.2} + \frac{(n-2)^4 x^4}{1.2.3.4} - \&c.\right) + \&c. \right\},$$

or

$$(\cos x)^n = \frac{1}{2^{n-1}} \left\{ \cos nx + n\cos(n-2)x \right.$$

$$\left. + \frac{n.(n-1)}{2}\cos(n-4)x + \&c. \right\}$$

(**148.**) In this formula for x put $\frac{\pi}{2} - y$; then

$$(\sin y)^n = \frac{1}{2^{n-1}}\left\{\cos\left(\frac{n\pi}{2} - ny\right) + n \cdot \cos\left(\frac{(n-2)\pi}{2} - (n-2)y\right)\right.$$

$$\left. + \frac{n \cdot (n-1)}{2} \cdot \cos\left(\frac{(n-4)\pi}{2} - (n-4)y\right) + \&c.\right\}$$

Let $n = 4p$; then $\cos\left(\frac{n\pi}{2} - ny\right) = \cos(2p\pi - ny)$

$$= \cos 2p\pi \cdot \cos ny + \sin 2p\pi \cdot \sin ny = \cos ny;$$

$$\cos\left(\frac{(n-2)\pi}{2} - (n-2)y\right) = \cos(2p-1)\pi \cdot \cos(n-2)y$$

$$+ \sin(2p-1)\pi \cdot \sin(n-2)y = -\cos(n-2)y, \&c.;$$

therefore, in this case, $(\sin y)^n = \frac{1}{2^{n-1}}\left\{\cos ny - n \cdot \cos(n-2)y\right.$

$$\left. + \frac{n \cdot (n-1)}{2} \cos(n-4)y - \&c.\right\}$$

Let $n = 4p + 2$; then in the same manner it is found, that

$$(\sin y)^n = \frac{1}{2^{n-1}}\left\{- \cos ny + n \cdot \cos(n-2)y\right.$$

$$\left. - \frac{n \cdot (n-1)}{2} \cos(n-4)y + \&c.\right\}$$

Let $n = 4p + 1$; $\cos\left(\frac{n\pi}{2} - ny\right) = \cos(2p + \frac{1}{2})\pi \cdot \cos ny$

$$+ \sin(2p + \tfrac{1}{2})\pi \cdot \sin ny = \sin ny;$$

$$\cos\left(\frac{(n-2)\pi}{2} - (n-2)y\right) = \cos(2p - \tfrac{1}{2})\pi \cdot \cos(n-2)y$$

$$+ \sin(2p - \tfrac{1}{2})\pi \cdot \sin(n-2)y = -\sin(n-2)y, \&c.;$$

and therefore in this case

$$(\sin y)^n = \frac{1}{2^{n-1}}\left\{\sin ny - n \cdot \sin(n-2)y\right.$$

$$\left. + \frac{n \cdot (n-1)}{2} \sin(n-4)y - \&c.\right\}$$

Let $n = 4p + 3$; then, in the same way,

$$(\sin y)^n = \frac{1}{2^{n-1}}\left\{- \sin ny + n \cdot \sin(n-2)y\right.$$

$$\left. - \frac{n \cdot (n-1)}{2} \cdot \sin(n-4)y + \&c.\right\}$$

(149.) When n is even, the last term in the expression for $(\cos x)^n$ and for $(\sin y)^n$ is

$$\frac{n\,.\,(n-1)\,\ldots\ldots\left(\frac{n}{2}+1\right)}{1\,.\,2\,\ldots\ldots\ldots\,\frac{n}{2}}\,.\,\frac{1}{2^n}=\frac{1}{2^n}\,.\,\frac{1\,.\,2\,.\,3\,\ldots\,(n-1)\,.\,n}{\left(1\,.\,2\,\ldots\,\frac{n}{2}\right)^2}$$

$$=\frac{1\,.\,2\,.\,3\,\ldots\,(n-1)\,.\,n}{(2\,.\,4\,\ldots\ldots\,n)^2}=\frac{1\,.\,3\,.\,5\,\ldots\,(n-1)}{2\,.\,4\,.\,6\,\ldots\ldots\,n}.$$

(150.) One of the principal uses of these expressions is the simplification of integrals taken between two values of x or y that differ by a circumference. Since $\int \cos px\,.\,dx$, as well as $\int \sin px\,.\,dx$ (p being an integer), always vanishes between two such values, it appears that through a whole circumference $\int (\cos x)^n\,.\,dx$ or $\int (\sin x)^n\,.\,dx$

is $=0$ when n is odd, and $=2\pi\,.\,\dfrac{1\,.\,3\,.\,5\,\ldots\,(n-1)}{2\,.\,4\,.\,6\,\ldots\ldots\,n}$ when n is even.

(151.) Since $\phi\,(\cos x)$ can generally be expanded in integral powers of $\cos x$, it can generally be expanded in cosines of multiples of x. This in most cases can be effected with the greatest ease by particular artifices, and especially by the use of the imaginary expression for $\cos x$, &c., as we proceed to show by examples.

(152.) Suppose $\tan\phi = n\tan\theta$; it is required to find a series for ϕ in terms of θ. If in $\tan\phi$ and $\tan\theta$ we put the values for $\sin\phi$, $\cos\phi$, &c., found in (141), we have

$$\frac{e^{\phi\sqrt{-1}}-e^{-\phi\sqrt{-1}}}{e^{\phi\sqrt{-1}}+e^{-\phi\sqrt{-1}}}=n\,.\,\frac{e^{\theta\sqrt{-1}}-e^{-\theta\sqrt{-1}}}{e^{\theta\sqrt{-1}}+e^{-\theta\sqrt{-1}}},$$

or
$$\frac{e^{2\phi\sqrt{-1}}-1}{e^{2\phi\sqrt{-1}}-1}=n\,.\,\frac{e^{2\theta\sqrt{-1}}-1}{e^{2\theta\sqrt{-1}}+1},$$

whence
$$e^{2\phi\sqrt{-1}}=e^{2\theta\sqrt{-1}}\,.\,\frac{1-\frac{n-1}{n+1}\,e^{-2\theta\sqrt{-1}}}{1-\frac{n-1}{n+1}\,e^{2\theta\sqrt{-1}}}.$$

Let $\dfrac{n-1}{n+1}=k$, and take the logarithms of both sides; then

$$2\phi\sqrt{-1}=2\theta\sqrt{-1}+\log\left(1-k\,.\,e^{-2\theta\sqrt{-1}}\right)-\log\left(1-k\,.\,e^{2\theta\sqrt{-1}}\right)$$

$$=2\theta\sqrt{-1}+k\left(e^{2\theta\sqrt{-1}}-e^{-2\theta\sqrt{-1}}\right)+\frac{k^2}{2}\left(e^{4\theta\sqrt{-1}}-e^{-4\theta\sqrt{-1}}\right)$$

$$+\frac{k^3}{3}\left(e^{6\theta\sqrt{-1}}-e^{-6\theta\sqrt{-1}}\right)+\&c.$$

Dividing by $2\sqrt{-1}$,

$$\phi = \theta + k \sin 2\theta + \frac{k^2}{2} \sin 4\theta + \frac{k^3}{3} \sin 6\theta + \&c.;$$

a theorem of great utility. The truth of the process is to be proved as in (147). In the same manner we might find a series if

$$\tan \phi = \frac{n \sin \theta}{1 - n \cos \theta},$$

n being less than 1.

(153.) To expand $(a^2 - 2ab\cos\theta + b^2)^n$ in a series proceeding by cosines of multiples of θ, b being less than a.

Since $2\cos\theta = e^{\theta\sqrt{-1}} + {}^{-\theta\sqrt{-1}}$, this expression

$$= \{(a - b\,e^{\theta\sqrt{-1}}) \,.\, (a - b\,e^{-\theta\sqrt{-1}})\}^n$$

$$= a^{2n} \,.\, \left(1 - \frac{b}{a}\,e^{\theta\sqrt{-1}}\right)^n . \left(1 - \frac{b}{a}\,e^{-\theta\sqrt{-1}}\right)^n.$$

Now $\left(1 - \frac{b}{a}\,e^{\theta\sqrt{-1}}\right)^n = 1 - n\,\frac{b}{a}\,e^{\theta\sqrt{-1}} + \frac{n\,.\,(n-1)}{1\,.\,2} \,.\, \frac{b^2}{a^2} \,.\, e^{2\theta\sqrt{-1}}$

$$- \frac{n\,.\,(n-1)\,.\,(n-2)}{1\,.\,2\,.\,3} \,.\, \frac{b^3}{a^3} \,.\, e^{3\theta\sqrt{-1}} + \&c.$$

$$\left(1 - \frac{b}{a}\,e^{-\theta\sqrt{-1}}\right)^n = 1 - n\,\frac{b}{a}\,e^{-\theta\sqrt{-1}} + \frac{n\,.\,(n-1)}{1\,.\,2} \,.\, \frac{b^2}{a^2}\,e^{-2\theta\sqrt{-1}}$$

$$- \frac{n\,.\,(n-1)\,.\,(n-2)}{1\,.\,2\,.\,3} \,.\, \frac{b^3}{a^3} \,.\, e^{-3\theta\sqrt{-1}} + \&c.$$

The product of these (observing that $e^{2\theta\sqrt{-1}} + e^{-2\theta\sqrt{-1}} = 2\cos 2\theta$, &c.)

$$= 1 + n^2\,\frac{b^2}{a^2} + \left(\frac{n\,(n-1)}{1\,.\,2}\right)^2 . \frac{b^4}{a^4} + \left(\frac{n\,(n-1)\,(n-2)}{1\,.\,2\,.\,3}\right)^2 . \frac{b^6}{a^6} + \&c.$$

$$- \left(n\,\frac{b}{a} + n \,.\, \frac{n\,.\,(n-1)}{1\,.\,2} \,.\, \frac{b^3}{a^3}\right.$$

$$\left. + \frac{n\,.\,(n-1)}{1\,.\,2} \,.\, \frac{n\,.\,(n-1)\,.\,(n-2)}{1\,.\,2\,.\,3} \,.\, \frac{b^5}{a^5} + \&c.\right) 2\cos\theta$$

$$+ \left(\frac{n\,.\,(n-1)}{1\,.\,2} \,.\, \frac{b^2}{a^2} + n \,.\, \frac{n\,.\,(n-1)\,.\,(n-2)}{1\,.\,2\,.\,3} \,.\, \frac{b^4}{a^4} + \&c.\right) 2\cos 2\theta$$

$$-\left(\frac{n\,.\,(n-1)\,.\,(n-2)}{1\,.\,2\,.\,3}\,.\,\frac{b^3}{a^3}+\&c.\right)2\cos 3\,\theta$$
$$+\ \&c.$$

Multiplying this by a^{2n} we have the series required.

(**154.**) To find $\log\,(1-n\cos\theta)$, n being less than 1, in a similar series.

Let $\qquad 1-n\cos\theta=(a-b\,e^{\theta\sqrt{-1}})\,(a-b\,e^{-\theta\sqrt{-1}})$
$$=a^2+b^2-2\,a\,b\cos\theta,$$
therefore $\qquad a+b=\sqrt{1+n},\ a-b=\sqrt{1-n}.$

$$\text{The log}=2\log a+\log\left(1-\frac{b}{a}\,e^{\theta\sqrt{-1}}\right)+\log\left(1-\frac{b}{a}\,e^{-\theta\sqrt{-1}}\right)$$
$$\left.\begin{aligned}=2\log a-\frac{b}{a}\,.\,e^{\theta\sqrt{-1}}-\frac{b^2}{2\,a^2}\,.\,e^{2\theta\sqrt{-1}}-\&c.\\ -\frac{b}{a}\,.\,e^{-\theta\sqrt{-1}}-\frac{b^2}{2\,a^2}\,.\,e^{-2\theta\sqrt{-1}}-\&c.\end{aligned}\right\}$$
$$=2\log a-\frac{2\,b}{a}\cos\theta-\frac{2\,b^2}{2\,a^2}\cos 2\,\theta-\&c.$$

And $\quad a=\tfrac{1}{2}\,(\sqrt{1+n}+\sqrt{1-n}),\ a^2=\tfrac{1}{2}\,(1+\sqrt{1-n^2});$
$$\frac{b}{a}=\frac{\sqrt{1+n}-\sqrt{1-n}}{\sqrt{1+n}+\sqrt{1-n}}=\frac{n}{1+\sqrt{1-n^2}};$$

whence $\qquad \log\,(1-n\cos\theta)=\log\dfrac{1+\sqrt{1-n^2}}{2}$
$$-\tfrac{2}{1}\left(\frac{n}{1+\sqrt{1-n^2}}\right)\cos\theta-\tfrac{2}{2}\left(\frac{n}{1+\sqrt{1-n^2}}\right)^2\cos 2\,\theta$$
$$-\tfrac{2}{3}\left(\frac{n}{1+\sqrt{1-n^2}}\right)^3\cos 3\,\theta-\&c.$$

(**155.**) To express $\sin x$ by a continued product.

We have seen in (145) that
$$x^{2n}-a^{2n}=(x-a)\left(x^2-2\,a\,x\cos\frac{\pi}{n}+a^2\right)$$
$$\left(x^2-2\,a\,x\,.\cos\frac{2\,\pi}{n}+a^2\right)\ldots\ldots\ (\overline{n-1}\text{ terms})\ \ldots\ldots\ (x+a);$$

dividing by $(x - a)$, $x^{2n-1} + x^{2n-2} . a + \&c. + a^{2n-1}$

$$= \left(x^2 - 2\,a\,x \cos\frac{\pi}{n} + a^2\right) \cdot \left(x^2 - 2\,a\,x\cos\frac{2\,\pi}{n} + a^2\right) \ldots .$$

$$\ldots . (\overline{n-1} \text{ terms}) \ldots . (x + a);$$

this is true if x differ from a, however small the difference may be. By making that difference very small, and making $a = 1$, we have this equation for the limit of that above;

$$2\,n = 2\left(1 - \cos\frac{\pi}{n}\right) 2\left(1 - \cos\frac{2\,\pi}{x}\right) \ldots . (\overline{n-1} \text{ terms}) \ldots . 2$$

$$= 2^{2n-1} . \sin^2\frac{\pi}{2\,n} . \sin^2\frac{2\,\pi}{2\,n} . \sin^2\frac{3\,\pi}{3\,n} \ldots . (\overline{n-1} \text{ terms}).$$

Again, let $x = 1 + \dfrac{z}{2\,n},\ a = 1 - \dfrac{z}{2\,n};$

then $x^2 + a^2 = 2 + 2\left(\dfrac{z}{2\,n}\right)^2;\ 2\,a\,x = 2 - 2\left(\dfrac{z}{2\,n}\right)^2,$

and the first equation becomes $\left(1 + \dfrac{z}{2\,n}\right)^{2n} - \left(1 - \dfrac{z}{2\,n}\right)^{2n}$

$$= \frac{2\,z}{2\,n} . 2\left\{\left(1 - \cos\frac{\pi}{n}\right) + \left(\frac{z}{2\,n}\right)^2 \cdot \left(1 + \cos\frac{\pi}{n}\right)\right\} \cdot$$

$$2\left\{\left(1 - \cos\frac{2\pi}{n}\right) + \left(\frac{z}{2n}\right)^2 \cdot \left(1 + \cos\frac{2\pi}{n}\right)\right\} \ldots (n-1 \text{ terms}) \ldots \times 2.$$

Or, since $1 - \cos\dfrac{\pi}{n} = 2\sin^2\dfrac{\pi}{2\,n}$, and $\dfrac{1 + \cos\dfrac{\pi}{n}}{1 - \cos\dfrac{\pi}{n}} = \operatorname{cotan}^2\dfrac{\pi}{2\,n}$,

we have $\left(1 + \dfrac{z}{2\,n}\right)^{2n} - \left(1 - \dfrac{z}{2\,n}\right)^{2n}$

$$= 2^{2n} . \sin^2\frac{\pi}{2\,n} . \sin^2\frac{2\,\pi}{2\,n} . \sin^2\frac{3\,\pi}{2\,n} \ldots . (\overline{n-1} \text{ terms}) \ldots . \frac{z}{2\,n} \cdot$$

$$\left\{1 + \left(\frac{z}{2\,n}\right)^2 \cot^2\frac{\pi}{2\,n}\right\} \cdot \left\{1 + \left(\frac{z}{2\,n}\right)^2 . \cot^2\frac{2\,\pi}{2\,n}\right\} \ldots . (\overline{n-1} \text{ terms}),$$

which the former equation reduces to $\left(1+\frac{z}{2n}\right)^{2n}-\left(1-\frac{z}{2n}\right)^{2n}$

$$=2z\left\{1+\left(\frac{z}{2n}\right)^2\cot^2\frac{\pi}{2n}\right\}\cdot\left\{1+\left(\frac{z}{2n}\right)^2\cot^2\frac{2\pi}{2n}\right\}\ ..\ (\overline{n-1}\text{ terms}).$$

Now suppose n indefinitely great;

since $\left(1+\frac{z}{2n}\right)^{2n}=1+2n\cdot\frac{z}{2n}+\frac{2n\cdot(2n-1)}{1\cdot 2}\cdot\frac{z^2}{4n^2}+\&c.,$

or $$=1+z+\frac{1-\frac{1}{2n}}{1\cdot 2}z^2+\&c.,$$

the limit of the first side is $2\left(z+\frac{z^3}{1\cdot 2\cdot 3}+\frac{z^5}{1\cdot 2\cdot 3\cdot 4\cdot 5}+\&c.\right)$;

since $\frac{z}{2n}\cdot\cot\frac{\pi}{2n}=\frac{z}{\pi}\cdot\frac{\frac{\pi}{2n}}{\tan\frac{\pi}{2n}}=$ ultimately $\frac{z}{\pi}$, the limit of the second side is $2z\left(1+\frac{z^2}{\pi^2}\right)\cdot\left(1+\frac{z^2}{4\pi^2}\right)\cdot\&c.$, indefinitely continued.

Dividing both by $2z$,

$$1+\frac{z^2}{1\cdot 2\cdot 3}+\frac{z^4}{1\cdot 2\cdot 3\cdot 4\cdot 5}+\&c.=\left(1+\frac{z^2}{\pi^2}\right)\cdot\left(1+\frac{z^2}{4\pi^2}\right)\cdot\&c.;$$

therefore, as in (147),

$$1-\frac{x^2}{1\cdot 2\cdot 3}+\frac{x^4}{1\cdot 2\cdot 3\cdot 4\cdot 5}-\&c.=\left(1-\frac{x^2}{\pi^2}\right)\cdot\left(1-\frac{x^2}{4\pi^2}\right)\cdot\&c.$$

and multiplying both sides by x,

$$\sin x=x\left(1-\frac{x^2}{\pi^2}\right)\cdot\left(1-\frac{x^2}{4\pi^2}\right)\cdot\left(1-\frac{x^2}{9\pi^2}\right)\cdot\&c.,\ \textit{ad infinitum}.$$

(156.) To express $\cos x$ by a continued product.

By (145), $x^{2n}+a^{2n}=\left(x^2-2ax\cos\frac{\pi}{2n}+a^2\right)\cdot$

$$\left(x^2-2ax\cos\frac{3\pi}{2n}+a^2\right)\cdot\&c.\text{ to }n\text{ terms.}$$

Let $x = 1$, $a = 1$; then

$$2 = 2\left(1 - \cos\frac{\pi}{2n}\right) . 2\left(1 - \cos\frac{3\pi}{2n}\right) \ldots . (n \text{ terms})$$

$$= 2^{2n} . \sin^2\frac{\pi}{4n} . \sin^2\frac{3\pi}{4n} \ldots (n \text{ terms}).$$

Again, let $x = 1 + \frac{z}{2n}$, $a = 1 - \frac{z}{2n}$; then as before

$$\left(1 + \frac{z}{2n}\right)^{2n} + \left(1 - \frac{z}{2n}\right)^{2n}$$

$$= 2^{2n} . \sin^2\frac{\pi}{4n} . \sin^2\frac{3\pi}{4n} \ldots . (n \text{ terms}) \ldots .$$

$$\times \left\{1 + \left(\frac{z}{2n}\right)^2 . \cot^2\frac{\pi}{4n}\right\} . \left\{1 + \left(\frac{z}{2n}\right)^2 . \cot^2\frac{3\pi}{4n}\right\} \ldots . (n \text{ terms}),$$

and the equation just found reduces this to

$$\left(1 + \frac{z}{2n}\right)^{2n} + \left(1 - \frac{z}{2n}\right)^{2n}$$

$$= 2\left\{1 + \left(\frac{z}{2n}\right)^2 \cot^2\frac{\pi}{4n}\right\}\left\{1 + \left(\frac{z}{2n}\right)^2 \cot^2\frac{3\pi}{4n}\right\} \ldots . (n \text{ terms});$$

and taking the limit of each side when n is indefinitely increased,

$$1 + \frac{z^2}{1 . 2} + \frac{z^4}{1 . 2 . 3 . 4} + \&c. = \left(1 + \frac{4z^2}{\pi^2}\right)\left(1 + \frac{4z^2}{9\pi^2}\right) . \&c.,$$

therefore $$1 - \frac{x^2}{1 . 2} + \frac{x^4}{1 . 2 . 3 . 4} - \&c.$$

$$= \left(1 - \frac{4x^2}{\pi^2}\right)\left(1 - \frac{4x^2}{9\pi^2}\right) . \&c. = \cos x.$$

(157.) Taking the differential coefficient with respect to x of the logarithm of the expression for $\cos x$ we find

$$\tan x = \frac{8x}{\pi^2 - 4x^2} + \frac{8x}{9\pi^2 - 4x^2} + \&c.$$

Similarly, from the expression for $\sin x$,

$$\cot x = \frac{1}{x} - \frac{2x}{\pi^2 - x^2} - \frac{2x}{4\pi^2 - x^2} + \&c.$$

(158.) The following theorems we shall find useful hereafter,

$$x^{2n} - 2x^n \cos\alpha + 1$$

$$= \left\{x^n - (\cos\alpha + \sqrt{-1} . \sin\alpha)\right\} . \left\{x^n - (\cos\alpha - \sqrt{-1} . \sin\alpha)\right\}.$$

If we solve the equation

$$x^n - (\cos\alpha + \sqrt{-1} \,.\, \sin\alpha) = 0,$$

we have $$x = (\cos\alpha + \sqrt{-1} \,.\, \sin\alpha)^{\frac{1}{n}};$$

the different values of which, as will be seen upon applying the theorems of (142) and (11), are

$$\cos\frac{\alpha}{n} + \sqrt{-1} \,.\, \sin\frac{\alpha}{n}; \quad \cos\frac{2\pi+\alpha}{n} + \sqrt{-1} \,.\, \sin\frac{2\pi+\alpha}{n}, \text{ \&c.}$$

and the factors of $x^n - (\cos\alpha + \sqrt{-1} \,.\, \sin\alpha)$ are therefore

$$x - \left(\cos\frac{\alpha}{n} + \sqrt{-1} \,.\, \sin\frac{\alpha}{n}\right),$$

$$x - \left(\cos\frac{2\pi+\alpha}{n} + \sqrt{-1} \,.\, \sin\frac{2\pi+\alpha}{n}\right), \text{ \&c.}$$

Similarly, the factors of $x^n - (\cos\alpha - \sqrt{-1} \,.\, \sin\alpha)$ are

$$x - \left(\cos\frac{\alpha}{n} - \sqrt{-1} \,.\, \sin\frac{\alpha}{n}\right),$$

$$x - \left(\cos\frac{2\pi+\alpha}{n} - \sqrt{-1} \,.\, \sin\frac{2\pi+\alpha}{n}\right), \text{ \&c.}$$

Combining the similar factors,

$$x^{2n} - 2x^n \,.\, \cos\alpha + 1 = \left(x^2 - 2x\cos\frac{\alpha}{n} + 1\right) .$$

$$. \left(x^2 - 2x\cos\frac{2\pi+\alpha}{n} + 1\right) . \left(x^2 - 2x \,.\, \cos\frac{4\pi+\alpha}{n} + 1\right), \text{ \&c.}$$

to n terms.

(159.) Now let $x = 1$; $x^{2n} - 2x^n \,.\, \cos\alpha + 1$ becomes $2 - 2\cos\alpha$ $= 4\sin^2\frac{\alpha}{2}$; $x^2 - 2x\cos\frac{\alpha}{n} + 1$ becomes $2 - 2\cos\frac{\alpha}{n} = 4\sin^2\frac{\alpha}{2n}$, &c., and the equation is changed to this;

$$4\sin^2\frac{\alpha}{2} = 4^n \,.\, \sin^2\frac{\alpha}{2n} \,.\, \sin^2\frac{2\pi+\alpha}{2n} \,.\, \sin^2\frac{4\pi+\alpha}{2n} \ldots\ldots (n \text{ terms}),$$

or $$\sin\frac{\alpha}{2} = 2^{n-1} \,.\, \sin\frac{\alpha}{2n} \,.\, \sin\frac{2\pi+}{2n} \,.\, \sin\frac{4\pi+\alpha}{2n} \ldots\ldots (n \text{ terms}).$$

Let $\frac{\alpha}{2n} = \beta$; then

$$\sin n\beta = 2^{n-1} \,.\, \sin\beta \,.\, \sin\left(\beta + \frac{\pi}{n}\right) \,.\, \sin\left(\beta + \frac{2\pi}{n}\right) \ldots\ldots (n \text{ terms}).$$

(160.) In the equation $x^{2n} - 2x^n \cdot \cos\alpha + 1$

$$= \left(x^2 - 2x \cdot \cos\frac{\alpha}{n} + 1\right)\left(x^2 - 2x \cdot \cos\frac{2\pi + \alpha}{n} + 1\right) \&c.,$$

the coefficient of x^{2n-1} must

$$= -2\left(\cos\frac{\alpha}{n} + \cos\frac{2\pi + \alpha}{n} + \cos\frac{4\pi + \alpha}{n} + \&c. \ (n \text{ terms})\right).$$

But this coefficient $= 0$; therefore, putting γ for $\frac{\alpha}{n}$,

$$\cos\gamma + \cos\left(\gamma + \frac{2\pi}{n}\right) + \cos\left(\gamma + \frac{4\pi}{n}\right) + \&c., \ (n \text{ terms}) = 0.$$

If n be even, this is an identical equation. If n be odd, the terms are all different, and observing that the cosine of an arc is the same as that of its defect from 2π, the equation, supposing γ less than $\frac{\pi}{n}$, may be put under this form,

$$\left.\begin{aligned} \cos\gamma + \cos\left(\frac{2\pi}{n} + \gamma\right) + \cos\left(\frac{4\pi}{n} + \gamma\right) + \&c. \\ + \cos\left(\frac{2\pi}{n} - \gamma\right) + \cos\left(\frac{4\pi}{n} - \gamma\right) + \&c. \end{aligned}\right\} = 0,$$

where each line is to be continued to that value of the arc which is next less than π. By transferring to the second side those terms that are negative, this is easily changed into the following,

$$\left.\begin{aligned} \cos\gamma + \cos\left(\frac{2\pi}{n} + \gamma\right) + \cos\left(\frac{4\pi}{n} + \gamma\right) + \&c. \\ + \cos\left(\frac{2\pi}{n} - \gamma\right) + \cos\left(\frac{4\pi}{n} - \gamma\right) + \&c. \end{aligned}\right\}$$

$$= \left\{\begin{aligned} \cos\left(\frac{\pi}{n} + \gamma\right) + \cos\left(\frac{3\pi}{n} + \gamma\right) + \&c. \\ + \cos\left(\frac{\pi}{n} - \gamma\right) + \cos\left(\frac{3\pi}{n} - \gamma\right) + \&c. \end{aligned}\right.$$

in which, γ being supposed less than $\frac{\pi}{2n}$, each series is to be continued till the angle reaches its greatest value next below 90°.

If n be made $= 5$, it will easily be seen that the last theorem of (49) is but a particular case of this.

(161.) In (124) and the following articles we explained a method of finding the corresponding small variations of parts of triangles. This may sometimes be abridged by the Differential Calculus. For if a, a function of c, receive the variation δa, in consequence of c receiving the variation δc, then

$$\delta a = \frac{d\,a}{d\,c}\,\delta c + \frac{d^2 a}{d\,c^2} \cdot \frac{(\delta c)^2}{1\,.\,2} + \&c.$$

If δc be very small, then $\delta a = \dfrac{d\,a}{d\,c}\,\delta c$ nearly.

If, however, $\dfrac{d\,a}{d\,c} = 0$, then $\delta a = \dfrac{d^2 a}{d\,c^2} \cdot \dfrac{(\delta c)^2}{1\,.\,2}$ nearly.

Thus, in the case of (128), $\sin a = \sin A \,.\, \sin c$;

$$\cos a \,.\, \frac{d\,a}{d\,c} = \sin A \,.\, \cos c, \text{ or } \frac{d\,a}{d\,c} = \frac{\sin A \,.\, \cos c}{\cos a};$$

therefore
$$\delta a = \frac{\sin A \,.\, \cos c}{\cos a}\,\delta c \text{ nearly.}$$

This is 0 when $c = \dfrac{\pi}{2}$; taking the second differential coefficient,

$$\cos a \,.\, \frac{d^2 a}{d\,c^2} - \sin a \,.\, \left(\frac{d\,a}{d\,c}\right)^2 = -\sin A \,.\, \sin c,$$

or
$$\cos a \,.\, \frac{d^2 a}{d\,c^2} = \frac{\sin a \,.\, \sin^2 A \,.\, \cos^2 c}{\cos^2 a} - \sin A \,.\, \sin c.$$

Make $c = \dfrac{\pi}{2}$; $a = A$; $\cos A \,.\, \dfrac{d^2 a}{d\,c^2} = -\sin A$; $\dfrac{d^2 a}{d\,c^2} = -\tan A$;

and $\delta a = -\tan A \,.\, \dfrac{(\delta c)^2}{2}$ nearly, as in (129).

(162.) This example sufficiently illustrates the use of this principle. For the cases in which the first differential coefficient does not vanish, and in which the neglect of the other terms will certainly introduce no error, it is convenient; but when a particular value makes the first differential coefficient vanish, or when it is necessary to examine the terms after the first, the method of (125) is generally preferable.

(163.) In our solutions of triangles, it will be remarked that we have frequently given several formulæ for the same case. The reason is, that in particular cases the value of an angle cannot at all, by the tables, be found exactly from its logarithmic sine or cosine; and in other cases it cannot be found exactly without much trouble. To provide, then, for all cases, several formulæ are sometimes necessary. We shall now show in what cases these difficulties occur.

(164.) The ratio of the small variation of any function of an arc to the variation of the arc being ultimately the differential coefficient, we shall have

$$\delta \,.\, \log \sin \theta = \frac{d \,.\, \log \sin \theta}{d\,\theta}\, \delta\,\theta \text{ nearly} = \text{M} \cot \theta \,.\, \delta\,\theta,$$

M being the modulus of common logarithms $= 0.43429448$. Now when θ is near 90°, $\cot \theta$ is very small, and a large variation of the arc is attended by a small variation of its log sine. A small error, then, in the log sine will produce a great error in the arc; or, if the tables be not carried to many decimals, the same log sine will correspond to several successive values of the arc. Consequently, an arc cannot be found accurately from its log sine when it is near 90°.

(165.) If now the arc be very small, M $\cot \theta$ becomes large; the second differential coefficient also ($= -$ M $\operatorname{cosec}^2 \theta$) is very great. It may happen, then, that the second differences of the log sines (of which the expression is $\frac{d^2 \,.\, \log \sin \theta}{d\,\theta^2}\, \delta\,\theta^2 + \text{\&c.}$) become large; and we must have the labour of interpolating by second differences. This, however, is commonly avoided by constructing tables for a few of the first degrees of the quadrant to every second, or to smaller intervals than the rest of the tables; $\delta\,\theta$ is thus made so small that the second differences are seldom sensible. But it is still better avoided by the use of a small table giving the logarithm of $\frac{\sin \theta}{\theta}$ for a few degrees.

For $\frac{\sin \theta}{\theta}$, by (140), $= 1 - \frac{\theta^2}{1\,.\,2\,.\,3} + \frac{\theta^4}{1\,.\,2\,.\,3\,.\,4\,.\,5} - \text{\&c.};$

$$\text{its logarithm} = - \text{M} \left(\frac{\theta^2}{6} + \frac{\theta^4}{180} + \text{\&c.}\right);$$

the differential coefficient of which, or $- \text{M} \left(\frac{\theta}{3} + \frac{\theta^3}{45} + \text{\&c.}\right)$, is very small when θ is small. And if $\theta = n''$, $\log \theta = \log n + \log 1''$, $= \log n + 4.6855749$, of which the first part can be found to any accuracy by common tables, and the second is constant; thus, when θ is small, $\log \sin \theta$ can be found accurately. The most convenient tables contain a table of $\log \frac{\sin \theta \times 1''}{\theta}$; let the number in this table corresponding to n'' be a, then $\log \sin n'' = a + \log n$.

(166.) Conversely from a given value of $\log \sin \theta$, θ when small is found with great ease. For subtracting from $\log \sin \theta$ the logarithm of 1″, or 4.6855749, we have, nearly, the log of the number of seconds, by which we find in the table the log $\frac{\sin\theta}{\theta}$ or the log $\frac{\sin\theta \times 1''}{\theta}$; and though the number of seconds is not theoretically exact, yet from the very slow variation of log . $\frac{\sin\theta}{\theta}$, the error in the result will not be sensible.

Then log true number of seconds $= \log\sin\theta - \log\frac{\sin\theta\times 1''}{\theta}$.

(167.) In the want of such tables, this method is convenient,

$$\frac{\sin\theta}{\theta} = 1 - \frac{\theta^2}{6} \text{ nearly} = \left(1 - \frac{\theta^2}{1\,.\,2}\right)^{\frac{1}{3}} = (\cos\theta)^{\frac{1}{3}} \text{ nearly};$$

therefore $$\log\frac{\sin\theta}{\delta} = \tfrac{1}{3}\log\cos\theta \text{ nearly.}$$

Hence, $$\log\sin\theta = \log\theta + \tfrac{1}{3}\log\cos\theta \text{ nearly}$$
$$= \log\theta - \tfrac{1}{3} \text{ arithmetical complement of } \log\cos\theta.$$

(168.) The same remarks in all respects apply to the tangent of a small arc.

$$\text{The series for the } \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\theta\left(1 - \frac{\theta^2}{6} + \&c.\right)}{1 - \frac{\theta^2}{2} + \&c.}$$
$$= \theta\left(1 + \frac{\theta^2}{3} + \&c.\right),$$

therefore $$\frac{\tan}{\theta} = 1 + \frac{\theta^2}{3} = \left(1 - \frac{\theta^2}{1\,.\,2}\right)^{-\frac{2}{3}},$$

and $$\log\tan\theta = \log\theta + \tfrac{2}{3} \text{ ar. comp. } \log\cos\theta \text{ nearly.}$$

These expressions can be used without sensible error till $\theta = 8°$. Since the differential coefficient of log tan θ $\left(= \frac{M}{\sin\theta\,.\,\cos\theta}\right)$ is never small, we can never meet with difficulties in the use of it like that mentioned in (164.)

(169.) In this way, then, we find that an arc cannot be determined accurately from its sine or cosecant when it is near 90°, from its cosine or secant when very small, or from its versed sine when near 180°; but from its tangent it can always be found with accuracy. Of the expressions, therefore, in (66) and (116), the first must not be used when $\frac{C}{2}$ is small or C is small; the second must not be used when C is near 180°, nor the fourth when C is near 90°. The third may always be used.

In (63) $\cos B = \frac{a}{c}$, which is inaccurate if B is small; but this expression may then safely be used, $\frac{1-\cos B}{1+\cos B}$, or $\tan^2 \frac{B}{2} = \frac{c-a}{c+a}$.

In (70), if B is near 90°, let $B = 90° \pm x$; then $\cos x = \frac{b}{a} \sin A$,

and $$\tan^2 \frac{x}{2} = \frac{1 - \frac{b}{a} \sin A}{1 + \frac{b}{a} \sin A}.$$

Now $\frac{b}{a}$ in all cases of difficulty will be greater than 1, and less than $\frac{1}{\sin A}$; let $\frac{a}{b} = \sin \theta$; then

$$\tan^2 \frac{x}{2} = \frac{\sin \theta - \sin A}{\sin \theta + \sin A} = \tan \frac{\theta - A}{2} \cdot \cot \frac{\theta + A}{2},$$

which can be calculated with accuracy.

In (105), if a and b be very small (a case which often occurs), c cannot be accurately found from that formula; we must therefore take $\tan A = \frac{\tan a}{\tan b}$, and $\tan c = \frac{\tan b}{\cos A}$, by which c is found to the greatest accuracy.

In (109), $\cos A = \frac{\tan b}{\tan c}$; if A be small,

$$\frac{1-\cos A}{1+\cos A} = \frac{\tan c - \tan b}{\tan c + \tan b}, \text{ or } \tan^2 \frac{A}{2} = \frac{\sin (c-b)}{\sin (c+b)},$$

which is not liable to inaccuracy.

In (118), if c should be near 180°, use this expression, $1 + \cos c$ $= 1 + \cos a \cdot \cos b + \sin a \cdot \sin b - \sin a \cdot \sin b \, (1 - \cos C)$,

or $$\cos^2 \frac{c}{2} = \cos^2 \frac{a-b}{2} - \sin a \cdot \sin b \cdot \sin^2 \frac{C}{2};$$

make $\sin a \cdot \sin b \cdot \sin^2 \frac{C}{2} = \sin^2 \theta$, then

$$\cos^2 \frac{c}{2} = \cos^2 \frac{a-b}{2} - \sin^2 \theta = \cos \left(\frac{a-b}{2} + \theta\right) \cdot \cos \left(\frac{a-b}{2} - \theta\right).$$

We have given, we believe, the most important cases; but in any others the same principle may easily be applied.

(170.) We shall conclude our remarks on this subject with the solution of the following problem: To find how far the tables are sufficiently exact. This will be done by giving to the arc the variation 1″, or any other, according to the degree of accuracy required, and finding at what limit the corresponding variation of the tabular numbers is equal to one unit in the last place of decimals. Thus, for log sines: by (164), the variation of log sin θ

$$\text{for } 1'' = 0.4343 \times \cot\theta \times 0.000004848.$$

If the tables be carried to 7 decimals, cot θ at the limit

$$= \frac{0.4343 \times 0.000004848}{0.0000001}; \text{ if to 10, } \cot\theta = \frac{0.4343 \times 0.000004848}{0.0000000001}.$$

The former gives $\theta = 87° \, 17'$; the latter gives $\theta = 89° \, 59' \, 50''$; and beyond these the tables of log sines cannot be trusted to seconds. The same principle may be applied to any other tables.

SECTION IX.

FORMULÆ PECULIAR TO GEODETIC OPERATIONS.

(171.) THE trigonometrical surveys, which have been carried on for the two objects of mapping an extensive country, and determining the figure and dimensions of the earth, afford the best exemplifications of most of the theorems both in plane and in spherical trigonometry. For some of the reductions, however, they require peculiar formulæ; these we shall give, after describing generally the course of operations.

(172.) The first part is the measurement of a base, for which a plain of four or five miles in extent is generally chosen; the line is measured with the most scrupulous exactness. In England, rods of deal, tubes of glass, and steel chains, have been used; the temperature being always noticed, and the proper correction applied for expansion. In the late surveys in France, the measuring rods consisted each of a rod of platina and a rod of brass, lying one upon the other, and connected at one extremity; the expansion of these metals being different, the difference of the expansions was observed, and the whole expansion of one bar found by a simple proportion.

Other bases are measured in different situations, called bases of verification, and their measure, compared with their length, as found by calculation, serves for a criterion of the correctness of the observations. Thus, for the French surveys of 1740, 17 bases were measured; but in the late surveys there, two only were used; and in the operations in Hindoostan, carried over a greater extent of country, five only were employed.

(173.) Proper situations for signals being selected, the country is divided into triangles by lines joining the stations; and the angles of the triangles, that is, the angles which two signals subtend, as seen from a third, are measured, (the first observation being made from the extremities of the base); and here the nature of the instruments used, modifies the calculation in a considerable degree. For the late French survey, repeating circles were employed, by means of which the angle between the two signals was observed; but since the signals are seldom seen exactly in the horizon, a calculation is necessary to find from this the horizontal angle. In England and India the horizontal angle was observed immediately by a theodolite.

(174.) In all the principal triangles each of the three angles is observed; and the error, if it is found from their sum that any exists, is divided among them in the most probable proportion. The sum of the errors in the nicer observations has seldom amounted to 2″. For the smaller triangles it is sufficient to measure two angles.

(175.) Beginning now with the measured base, we have the length of the base and the observed angles at its extremities, to determine the distance of a signal from its extremities, and the angle at the signal; that is, we have one side and the two adjacent angles to determine the other parts. Thus we determine A C, B C, (fig. 24). Similarly, AD, BD, are found; then C D is determined. Then in the triangle C E D we have similar data. And this process we extend to any number of triangles, till we arrive at a base of verification.

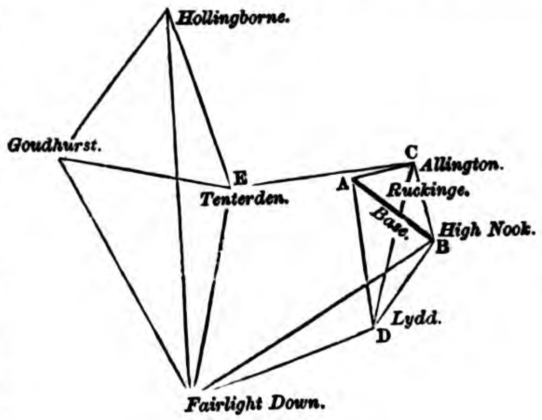

Fig. 24.

(176.) It is generally thought proper to choose the stations such that the sides of the triangles are greater than 10 and less than 20 miles. In the English survey, however, the distance from Beachy-Head to Dunnose, which formed one side of a triangle, is more than $64\frac{1}{4}$ miles. And in the extension of French survey to Spain, to connect Iviza with the continent, a triangle was formed, of which one side was nearly 100 miles. The calculations are verified either by comparing the calculated length of a base of verification with the measured length, or by comparing the distance between two signals as calculated from two chains of triangles, beginning either from the same base or from different bases. Thus, in England by a series of triangles, extending more than 200 miles, from Dunnose in the Isle of Wight to Clifton in Yorkshire, it was found that the error in a line of 22 miles does not exceed six feet. And in some of the English bases of

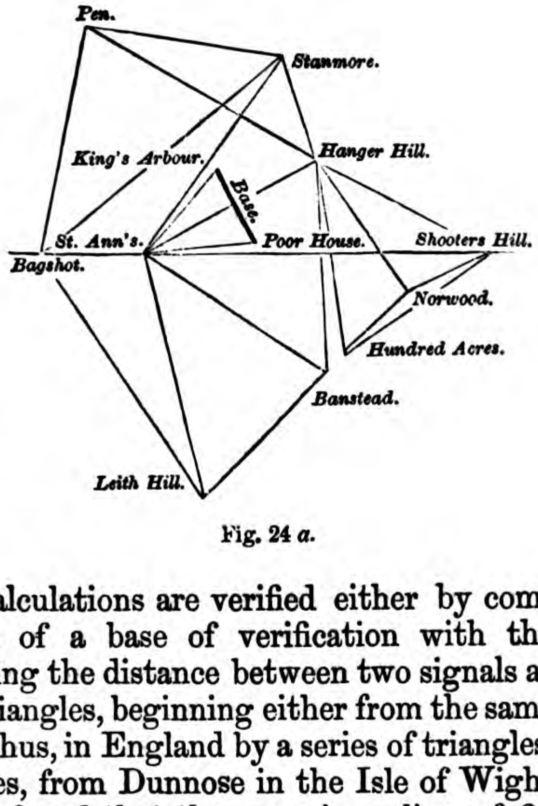

Fig. 24 a.

verification of four or five miles in length, the difference between the computed and measured lengths has not exceeded one or two inches.

(177.) The latitudes and longitudes of the principal stations (those of one being known) are then determined accurately, and those of the minor objects which have been observed by a more expeditious method. This is for the purpose of mapping; if it is intended to ascertain the length of a degree of latitude, the distance of two places in the direction of the meridian must be ascertained, and the latitude of each must be observed. This was the object of the late French survey; their purpose being to determine the length of the terrestrial quadrant, of which the 10,000,000th part, or *metre*, was made the standard of linear measure. For the determination of a degree of longitude (a calculation which implies the spheroidal form of the earth) methods are used of which it would be foreign to our purpose to treat.

(178.) This is a general explanation of the usual process: we shall now give the mathematical theorems connected with it. In the French bases, the line measured was not straight, but consisted of two parts, as a and b, (fig. 25), forming a small angle θ, (when largest it was 49′).

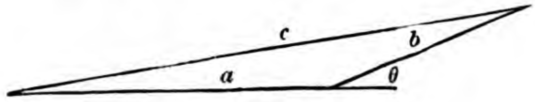

Fig. 25.

To find the correction,

$$c^2 = a^2 + b^2 - 2\,a\,b \,.\, \cos(\pi - \theta) = a^2 + b^2 + 2\,a\,b\,\cos\theta;$$

but since θ is very small, $\cos\theta = 1 - \dfrac{\theta^2}{2}$ nearly; therefore

$$c^2 = (a+b)^2 - a\,b\,.\,\theta^2, \text{ and } c = a + b - \frac{a\,b}{2\,(a+b)}\,\theta^2 \text{ nearly},$$

or the correction is $\dfrac{a\,b}{2\,(a+b)}\,\theta^2$.

If $\theta = n$ seconds $= n \times 0{,}000004848$,

$$\text{the correction} = \frac{a\,b\,.\,n^2}{a+b} \times 0{,}00000000001175.$$

(179.) Supposing the three angles of a triangle observed, and one side, as a, known, To find its figure, that the lengths of the other sides may be least affected by the errors of observation. Let A be the observed angle opposite to a, B and C the angles adjacent, and b and c the sides opposite to them. Suppose the errors of A, B, and C, to be δ A, δ B, and δ C; then, as the sum of the angles (if erroneous) is supposed to be corrected by altering each of the angles by the same quantity, $\delta\,\mathrm{A} = -(\delta\,\mathrm{B} + \delta\,\mathrm{C})$. Then the true value of c is

$$a\,.\,\frac{\sin(\mathrm{C} + \delta\,\mathrm{C})}{\sin(\mathrm{A} - \delta\,\mathrm{B} - \delta\,\mathrm{C})};$$

but $$\sin (C + \delta C) = \sin C \,.\, \cos \delta C + \cos C \,.\, \sin \delta C$$
$$= \sin C + \delta C \,.\, \cos C \text{ nearly};$$

putting a similar expression for the denominator, and observing that $\sin B = \sin (\pi - B) = \sin (A + C) = \sin A \cos C + \cos A \,.\, \sin C$,

we find $$c = a\left(\frac{\sin C}{\sin A} + \frac{\sin B}{\sin^2 A}\,\delta C + \frac{\sin C \,.\, \cos A}{\sin^2 A}\,\delta B\right),$$

or the error of c is $a\left(\frac{\sin B}{\sin^2 A}\,\delta C + \frac{\sin C \,.\, \cos A}{\sin^2 A}\,\delta B\right)$.

Similarly, the error of b is $a\left(\frac{\sin C}{\sin^2 A}\,\delta B + \frac{\sin B \,.\, \cos A}{\sin^2 A}\,\delta C\right)$.

Now, it is impossible to assign exactly the chances of the errors δB, δC, and δA, or $-(\delta B + \delta C)$, and our reasoning must therefore be vague. It is evident, however, that $\sin A$ must not be small; it is largest when $A = 90°$. But it is equally evident, that there is a greater chance that the signs of δB and δC are different, than that they are the same; since in the three pairs that we can form of δA, δB, δC, two will have errors of different signs, and one will have errors of the same sign. And if δB and δC have different signs, the errors of b and c will be diminished by giving $\cos A$ a positive value. A therefore ought to be less than 90°; and if δB and δC are probably not very different, B and C should be nearly equal. These conditions will be satisfied by a triangle differing not much from an equilateral triangle.

(180.) If two angles only, A and B, be observed, the expression for the errors will be as above; but we have now no reason to think them of different signs rather than of the same sign. In this case, then, we shall probably have our errors smallest, if $\cos A = 0$, or $A = 90°$; the remaining angle of the triangle ought therefore to be as nearly as possible a right angle.

(181.) The elevations or depressions of signals being small, the correction to be applied to their measured angular distance in order to obtain the horizontal angle is thus found. Suppose O A, O B, (fig. 26), to be the directions in which two signals are seen from O; the angle A O B is measured. If a sphere be supposed described about O as centre, and if through Z the point vertical to O great circles O A C, O B D, be drawn, and C O D be the horizontal plane, then C O D or Z, since Z is the pole of C D, is the horizontal angle required.

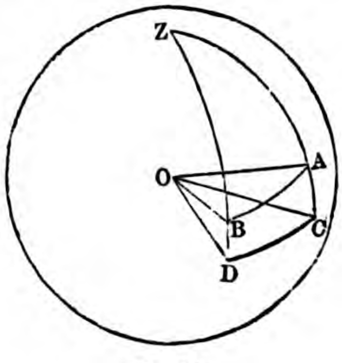

Fig. 26.

Now $$\cos Z = \frac{\cos AB - \cos ZA . \cos ZB}{\sin ZA . \sin ZB}.$$

Let $AB = D$, $Z = D + x$; $\cos Z = \cos D . \cos x - \sin D . \sin x$

$= \cos D - \sin D . x$ nearly, let $AC = h$, $BD = h'$, then

$\cos h = 1 - \frac{h^2}{2}$, $\cos h' = 1 - \frac{h'^2}{2}$, $\sin h = h$, $\sin h' = h'$, nearly;

and the equation becomes

$$\cos D - x \sin D = \frac{\cos D - h h'}{1 - \frac{h^2}{2} - \frac{h'^2}{2}} \text{ nearly}$$

$$= \cos D - h h' + \frac{\cos D}{2}(h^2 + h'^2);$$

therefore $$x = \frac{h h'}{\sin D} - \frac{\cos D}{2 \sin D}(h^2 + h'^2).$$

Let $$h + h' = p;\ h - h' = q;$$

therefore $$h h' = \frac{p^2 - q^2}{4};\ h^2 + h'^2 = \frac{p^2 + q^2}{2},$$

and $$x = \frac{1}{4}\left(\frac{p^2 - q^2}{\sin D} - \frac{(p^2 + q^2)\cos D}{\sin D}\right)$$

$$= \frac{1}{4}\left(p^2 \frac{1 - \cos D}{\sin D} - q^2 \frac{1 + \cos D}{\sin D}\right)$$

$$= \frac{1}{4}\left(p^2 \tan \frac{D}{2} - q^2 \cot \frac{D}{2}\right).$$

For observations with the theodolite, this is not necessary.

(182.) The horizontal angles being found, all our triangles are converted into spherical triangles, the sides of which are small compared with the radius of the sphere. For the solution of these triangles, three different methods are used. The first is to solve them as spherical triangles, taking for the sines of the sides the expressions in (165) and (167). Knowing nearly the radius of the earth, the angle subtended at the centre by an arc of given length is known, and hence $\log \frac{\sin a}{a}$ can be taken from a table where a is expressed in feet or toises; adding $\log a$, $\log \sin a$ is found. This method is, by Delambre, preferred to the others. The second is to find from the angles of the spherical triangles the angles formed by their chords, and to solve this as a plane triangle. Let C be one spherical angle, $C - x$ the angle contained by the chords, then

$$\cos (C - x) = \frac{(\text{chord of } a)^2 + (\text{chord of } b)^2 - (\text{chord of } c)^2}{2 \text{ chord of } a \cdot \text{chord of } b}$$

$$= \frac{\sin^2 \frac{a}{2} + \sin^2 \frac{b}{2} - \sin^2 \frac{c}{2}}{2 \sin \frac{a}{2} \cdot \sin \frac{b}{2}}.$$

But
$$\cos C = \frac{\cos c - \cos a \cdot \cos b}{\sin a \cdot \sin b}$$

$$= \frac{1 - 2 \sin^2 \frac{c}{2} - \left(1 - 2 \sin^2 \frac{a}{2}\right)\left(1 - 2 \sin^2 \frac{b}{2}\right)}{4 \sin \frac{a}{2} \cos \frac{a}{2} \sin \frac{b}{2} \cos \frac{b}{2}}$$

$$= \frac{\sin^2 \frac{a}{2} + \sin^2 \frac{b}{2} - \sin^2 \frac{c}{2}}{2 \sin \frac{a}{2} \sin \frac{b}{2} \cos \frac{a}{2} \cos \frac{b}{2}} - \frac{\sin \frac{a}{2} \cdot \sin \frac{b}{2}}{\cos \frac{a}{2} \cdot \cos \frac{b}{2}};$$

therefore $\cos (C - x) = \cos \frac{a}{2} \cdot \cos \frac{b}{2} \cos C + \sin \frac{a}{2} \cdot \sin \frac{b}{2}$

$$= \cos C + x \sin C;$$

therefore $x \sin C = \sin \frac{a}{2} \cdot \sin \frac{b}{2} - \left(1 - \cos \frac{a}{2} \cdot \cos \frac{b}{2}\right) \cos C$

$$= \frac{a\,b}{4} - \frac{a^2 + b^2}{8} \cos C.$$

Let $a + b = e$, $a - b = f$; therefore $a^2 + b^2 = \frac{e^2 + f^2}{2}$, $a\,b = \frac{e^2 - f^2}{4}$;

and
$$x \sin C = \frac{e^2 - f^2}{16} - \frac{e^2 + f^2}{16} \cos C,$$

or
$$x = \frac{1}{16}\left(e^2 \frac{1 - \cos C}{\sin C} - f^2 \cdot \frac{1 + \cos C}{\sin C}\right)$$

$$= \frac{1}{16}\left(e^2 \tan \frac{C}{2} - f^2 \cot \frac{C}{2}\right).$$

All these expressions suppose the angles to be expressed in numbers, considering the radius as 1; if $e = n$ feet, then for e we must put $\frac{n}{\text{number of feet in radius}}$; if $x = m$ seconds, for x we must put $m \times 0{,}000004848$. This method was used in the English surveys.

(**183**.) The principle of the third method is, by applying a correction to the angles of the spherical triangle to treat it as a plane triangle. Let a, b, c, be the sides to radius r; then

$$\cos C = \frac{\cos\frac{c}{r} - \cos\frac{a}{r}\cos\frac{b}{r}}{\sin\frac{a}{r}\,.\,\sin\frac{b}{r}}$$

$$= \frac{1 - \frac{c^2}{2\,r^2} + \frac{c^4}{24\,r^4} - \left(1 - \frac{a^2}{2\,r^2} + \frac{a^4}{24\,r^4}\right)\left(1 - \frac{b^2}{2\,r^2} + \frac{b^4}{24\,r^4}\right)}{\frac{a\,b}{r^2}\left(1 - \frac{a^2}{6\,r^2}\right)\left(1 - \frac{b^2}{6\,r^2}\right)} \text{ nearly}$$

$$= \frac{a^2 + b^2 - c^2 - \frac{1}{12\,r^2}\left\{2\,a^2 b^2 + 2\,a^2 c^2 + 2\,b^2 c^2 - a^4 - b^4 - c^4\right\}}{2\,a\,b}.$$

But if $C - x$ be the angle in the triangle considered as plane, then

$$\cos(C - x), \text{ or } \cos C + x \sin C = \frac{a^2 + b^2 - c^2}{2\,a\,b};$$

therefore $x \sin C = \dfrac{1}{24\,r^2\,a\,b}\left\{2\,a^2 b^2 + 2\,a^2 c^2 + 2\,b^2 c^2 - a^4 - b^4 - c^4\right\}$.

The part within the brackets $= 4\,a^2\,b^2 - (a^2 + b^2 - c^2)^2$

$$= \{2\,a\,b + (a^2 + b^2 - c^2)\}.\{2\,a\,b - (a^2 + b^2 - c^2)\}$$

$$= \{(a + b)^2 - c^2\}.\{c^2 - (a - b)^2\}$$

$$= (a + b + c)\,(a + b - c)\,(a + c - b)\,(b + c - a)$$

$$= 16\ (\text{area of triangle})^2.$$

But $a\,b \sin C = 2\,.\,\text{area}$; therefore $x = \dfrac{\text{area of triangle}}{3\,r^2}$;

or, if $x = n$ seconds, $n = \dfrac{\text{area of triangle}}{3\,r^2 \times 0{,}000004848}$.

This is Legendre's theorem. If the area of the triangle be found in feet, the logarithm of the divisor is 9,8038940, a degree on the earth's surface being considered = 365155 feet. This is due to General Roy. The dimensions of the triangle are always known accurately enough to find the area with sufficient exactness. The correction is the same for each of the angles; it is therefore one-third of the excess of the sum of the three angles above 180°, commonly called the spherical excess. The spherical excess seldom amounts to 5″; in the largest triangle joining Iviza with the coast of Spain it amounted however to 39″.

(184.) The sides and angles of the triangles being found by some of these methods, and the relative situation of the signals being found, it is necessary to determine the angle which some one of the lines makes with the meridian. In the English surveys this was done by observing with the theodolite the horizontal angle between a signal and the pole-star, at the time when it was known to be at its greatest azimuth. Let Z (fig. 27) be the zenith, P the pole, S the pole-star, Z S a great circle. Then cot Z . sin Z P S = cot S P . sin Z P — cos Z P S . cos Z P. Suppose a small error in the time, this would create a small error in the angle Z P S. Now, as in (131), we find that the simplified expression for the error of Z vanishes when cos S is 0, or S is a right angle. Returning then to the original expression, and observing that cos Z P = cos Z . cos P; and putting for cot $(Z + \delta Z)$, &c., their approximate values, we find at length

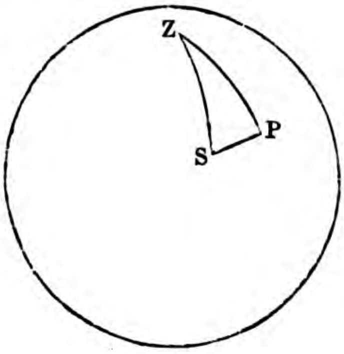

Fig. 27.

$$\delta Z = \frac{\sin Z . \cos Z}{\sin^2 P} . \frac{(\delta P)^2}{2}.$$

Now with the pole-star sin Z is small, and δP very small; hence a small error in time will not produce a sensible error in the azimuth.

(185.) In the French surveys the azimuth was found by observing the angle between the signal and the sun when near the horizon; also by taking the angular distance of the signal from the pole-star when nearest to the signal, or farthest from it. To allow the observations to be repeated, a correction was investigated not very dissimilar to that of the last article, to be applied to the observations made when the pole-star was a little removed from the point nearest to, or farthest from, the signal. From this distance the azimuth is found by a right-angled spherical triangle. But in Spain, a transit instrument being adjusted to a mark nearly in the meridian, the intervals of the transits of different stars were observed: comparing these intervals with those that ought to have been observed in the meridian, the azimuth of the mark was determined by a formula common in practical astronomy. From this the azimuth of any signal was easily found.

(186.) The direction of one side being known, we have now to solve this problem. Given P A (fig. 28), the colatitude of A, and the angle P A B, and the length of A B; to find P B the colatitude of B, and the angle B, and the difference of longitude P; A B being small (seldom = 1°). Here

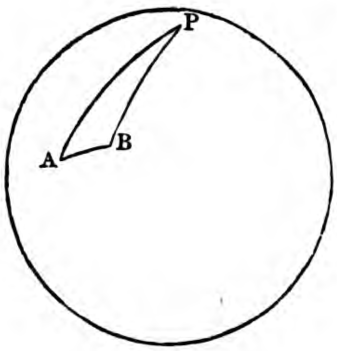

Fig. 28.

$$\cos B P = \cos A P . \cos A B$$
$$+ \sin A P . \sin A B . \cos A;$$

let $B P = A P - x$; then

$$\cos (A P - x) - \cos A P, \text{ or } 2 \sin \left(A P - \frac{x}{2}\right) . \sin \frac{x}{2}$$
$$= \sin A P . \sin A B . \cos A - \cos A P (1 - \cos A B)$$
$$= A B . \sin A P . \cos A - \cos A P . \frac{A B^2}{2} \text{ nearly};$$

$$\text{therefore } \sin \frac{x}{2} = \frac{A B \quad \sin A P . \cos A - \frac{A B^2}{2} \cos A P}{2 \sin \left(A P - \frac{x}{2}\right)}.$$

An approximate value of $\frac{x}{2}$ is $\frac{A B . \cos A}{2}$; substituting this in the denominator,

$$x = 2 \sin \frac{x}{2} \text{ nearly} = A B \cos A - \frac{\cot A P . \sin^2 A}{2} A B^2.$$

If greater accuracy is desired, this may be again substituted in the denominator; then $A B^3$ must be taken in the numerator; and observing that $\frac{x}{2} = \sin \frac{x}{2} + \frac{1}{6} \left(\sin \frac{x}{2}\right)^3$ nearly,

$$x = A B \cos A - \frac{\cot A P . \sin^2 A}{2} A B^2$$
$$+ \frac{(1 + 3 \cot^2 A P) \sin^4 A . \cos A}{6} A B^3.$$

Then $\sin P = \dfrac{\sin A B \,.\, \sin A}{\sin P B}$, and $\sin B = \dfrac{\sin A P \,.\, \sin A}{\sin P B}$.

Or, if a series be preferred,

$$P = \frac{A B \,.\, \sin A}{\sin P A} - \frac{A B^2 \,.\, \sin A \,.\, \cos A \,.\, \cot P A}{\sin P A} - \&c.;$$

$$B = 180° - A + \frac{A B \,.\, \sin A \,.\, \sin \dfrac{P A + P B}{2}}{\sin P B}.$$

(187.) For the points of less consequence, which have been observed from two stations, the distances being found considering the triangles as plane, the value $x = A B \cos \theta$ is sufficiently accurate; and then $P = \dfrac{A B \,.\, \sin A}{\sin P A}$ nearly.

These are the principal formulæ of trigonometry that are used for surveys on a large scale. We have treated of them at some length, as we know not any book in the English language in which any account of them is to be found. We have confined ourselves to what appeared to be strictly connected with the subject of this treatise; for the explanation of the methods used in different hypotheses of the figure of the earth, and for the results deduced from them, we refer to treatises on the figure of the earth.

SECTION X.

ON THE CONSTRUCTION OF TRIGONOMETRICAL TABLES.

(188.) THE construction of tables naturally divides itself into two parts: the first is, the determination of values of the function to be tabulated for certain values of the arc, at large intervals; the second is, the filling up of the tables by inserting the values included between these. In this order we propose to consider the formation of tables of the values of trigonometrical lines and their logarithms.

(189.) The method which first suggests itself for the determination of natural sines, is to take some arc whose sine and cosine are known, (as 30°, 45°, 18°, 54°, &c.) and determine the cosine of half the arc by the formula

$$\cos a = \sqrt{\frac{1 + \cos 2\,a}{2}},$$

and after repeated applications of it to determine the sine by the form

$$\sin a = \sqrt{\frac{1 - \cos 2\,a}{2}}.$$

Or the sine and cosine may be determined by the formula

$$\sin a = \tfrac{1}{2}\left\{\sqrt{1 + \sin 2\,a} - \sqrt{1 - \sin 2\,a}\right\},$$

$$\cos a = \tfrac{1}{2}\left\{\sqrt{1 + \sin 2\,a} + \sqrt{1 - \sin 2\,a}\right\}.$$

This method, when $2\,a$ is small, is more accurate than the former. For when $\frac{1 - \cos 2\,a}{2}$ is very small $= v$, suppose x to be the error to which it is liable, or the value of the figures rejected; then its square root will be liable to the error $\frac{x}{2\sqrt{v}}$ nearly, which, when v is small, is very considerable. On the contrary, in the other method, $1 + \sin 2\,a$, and $1 - \sin 2\,a$, being nearly $= 1$, upon extracting their roots we are not liable to the same error. In this manner find the sine of $\frac{30^\circ}{2^{11}} = 52'' \, 44''' \, 3'''' \, 45'''''$. Now by observation of the sines of this arc, and of the double of this arc, it will be seen that the sines of small arcs are nearly as the arcs; and therefore $52'' \, 44''' \, 3'''' \, 45''''' : 1' :: \text{sine found} : \text{sine of } 1'$. From this the cosine of $1'$ is found; and the sines and cosines of $2'$, $3'$, $4'$, &c. are found by the formulæ of (38).

(190.) But the same thing may be done in this manner, with fewer (though more laborious) operations, and without the proportion used in the last article. It was found that $\sin 5a = 5 \sin a - 20 \sin^3 a + 16 \sin^5 a$; conversely, the solution of the equation $5x - 20x^3 + 16x^5 = \sin 5a$ will give the value of $\sin a$. Thus, from $\sin 15^\circ$ (found by bisection) we may by approximation find $\sin 3^\circ$. Again, $\sin 3b = 3 \sin b - 4 \sin^3 b$; solving this equation we have the value of $\sin b$ from $\sin 3b$, and therefore from $\sin 3^\circ$ we find $\sin 1^\circ$. By a repetition of the same operations we descend to $\sin 30'$, $\sin 15'$, $\sin 3'$, $\sin 1'$; and then ascend as before. In the same way we might have begun from 18°, or any arc whose sine is known.

(191.) But in a process of this kind, where an error in the calculation of one number would affect all the following ones, it is clearly desirable to compute independently some numbers in the series at convenient intervals to serve as verifications for the rest. Thus, from $\sin 30^\circ$ we may by trisection find $\sin 10^\circ$; from this we get $\cos 10^\circ$ or $\sin 80^\circ$; then $\sin 20^\circ = 2 \sin 10^\circ \, . \, \cos 10^\circ$ is found; then since

$$\sin (60^\circ + A) - \sin (60^\circ - A) = \sin A,$$

we have

$$\sin 80^\circ - \sin 40^\circ = \sin 20^\circ,$$

whence $\sin 40^\circ$ is found; thence $\sin 50^\circ$ or $\cos 40^\circ$ is found; and

$$\sin 70^\circ = \sin 50^\circ + \sin 10^\circ.$$

The sines for every 10° of the quadrant being found, those of every degree should then be calculated as verifications for those of every minute, &c. The following is the best method of performing these calculations:

$$\sin (n+1) b = 2 \cos b \, . \, \sin n b - \sin (n-1) b,$$

therefore

$$\sin (n+1) b - \sin nb = \sin nb - \sin (n-1) b - (2 - 2 \cos b) \sin nb.$$

But $\quad \sin(n+1)b - \sin nb =$ difference of $\sin nb$;

$$\sin nb - \sin(n-1)b = \text{the preceding difference};$$

hence the difference is less than the preceding difference by

$$(2 - 2\cos b)\sin nb, \text{ or } 4\sin^2\frac{b}{2}\,.\,\sin nb;$$

that is, the second difference is $-4\sin^2\frac{b}{2}\,.\,\sin nb$. Now, since $\sin nb$ is already found, this can be calculated; and the operation will not be long, for the multiplier $4\sin^2\frac{b}{2}$ being the same every time, a table of its products by the 9 digits may be prepared. Thus then we have

$$\sin 12^\circ - \sin 11^\circ = \sin 11^\circ - \sin 10^\circ - 4\sin^2 30'\,.\,\sin 11^\circ, \text{ \&c.}$$

In this way the sines for every degree may be found; if the values for sin 10°, sin 20°, &c. are not the same as those found before, it shows that there is some error in the computation.

(192.) But the natural sines for these arcs, at least for 10°, 20°, &c., or more conveniently for 9°, 18°, &c., may be calculated independently thus. We found for $\sin x$ the series

$$x - \frac{x^3}{1\,.\,2\,.\,3} + \frac{x^5}{1\,.\,2\,.\,3\,.\,4\,.\,5} - \text{\&c.},$$

let $x = \frac{m}{n}\,.\,\frac{\pi}{2}$; then $\frac{\pi}{2}$ being found by the differential calculus to $= 1{,}570796326794897$, we have $\sin\frac{m\pi}{2n} =$

$$\begin{aligned}
&\frac{m}{n} \times 1{,}5707963267948 97 && -\frac{m^3}{n^3} \times 0{,}645964097506246\\
&+\frac{m^5}{n^5} \times 0{,}079692626246167 && -\frac{m^7}{n^7} \times 0{,}004681754135319\\
&+\frac{m^9}{n^9} \times 0{,}000160441184787 && -\frac{m^{11}}{n^{11}} \times 0{,}000003598843235\\
&+\frac{m^{13}}{n^{13}} \times 0{,}000000056921729 && -\frac{m^{15}}{n^{15}} \times 0{,}000000000668804\\
&+\frac{m^{17}}{n^{17}} \times 0{,}000000000006067 && -\frac{m^{19}}{n^{19}} \times 0{,}000000000000044
\end{aligned}$$

Similarly, as

$\cos x = 1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \&c.$ we have $\cos . \frac{m}{n} . \frac{\pi}{2} =$

$$
\begin{array}{llll}
 & 1{,}000000000000000 & - \frac{m^2}{n^2} & \times 1{,}233700550136170 \\
+ \frac{m^4}{n^4} & \times 0{,}253669507901048 & - \frac{m^6}{n^6} & \times 0{,}020863480763353 \\
+ \frac{m^8}{n^8} & \times 0{,}000919260274839 & - \frac{m^{10}}{n^{10}} & \times 0{,}000025202042373 \\
+ \frac{m^{12}}{n^{12}} & \times 0{,}000000471087478 & - \frac{m^{14}}{n^{14}} & \times 0{,}000000006386603 \\
+ \frac{m^{16}}{n^{16}} & \times 0{,}000000000065660 & - \frac{m^{18}}{n^{18}} & \times 0{,}000000000000529 \\
+ \frac{m^{20}}{n^{20}} & \times 0{,}000000000000003 & &
\end{array}
$$

The cosine of an arc being the sine of its complement, $\frac{m}{n}$ will never exceed $\frac{1}{2}$, and a few terms of these series will give the natural sines with great ease to 15 decimals.

(193.) When the sines for every degree are calculated, they should be verified; and for this purpose the last equation of (160) will be found extremely useful. By giving to γ and n different values, we may with great ease examine the accuracy of as many calculated sines as we wish.

(194.) The sines for degrees being found, those for smaller divisions, as minutes, are generally found by differences. And a remarkable relation between the differences of successive orders enables us to determine the differences with which we must begin our table, from knowing the two first of them. Let it be supposed that the arc x is formed by successive additions of h; then

$$\Delta \sin x = \sin (x + h) - \sin x = 2 \sin \frac{h}{2} . \cos \left(x + \frac{h}{2}\right);$$

$$\Delta^2 \sin x = 2 \sin \frac{h}{2} \left\{\cos \left(x + \frac{3h}{2}\right) - \cos \left(x + \frac{h}{2}\right)\right\}$$

$$= - 4 \sin^2 \frac{h}{2} . \sin (x + h);$$

and, consequently, $\Delta^2 \sin (x - h) = - 4 \sin^2 \frac{h}{2} \sin x$.

Hence, $$\Delta^4 . \sin (x - h) = - 4 \sin^2 \frac{h}{2} \Delta^2 \sin x$$

$$= 16 \sin^4 \frac{h}{2} . \sin (x + h),$$

and, therefore, $$\Delta^4 \sin (x - 2h) = 16 \sin^4 \frac{h}{2} . \sin x.$$

Similarly, $$\Delta^6 \sin (x - 3h) = - 64 \sin^6 \frac{h}{2} . \sin x, \text{ \&c.}$$

Also $$\Delta^3 \sin x = - 8 \sin^3 \frac{h}{2} . \cos \left(x + \frac{3h}{2}\right),$$

therefore $$\Delta^3 . \sin (x - h) = - 8 . \sin^3 \frac{h}{2} . \cos \left(x + \frac{h}{2}\right);$$

similarly, $$\Delta^5 \sin (x - 2h) = 32 \sin^5 \frac{h}{2} . \cos \left(x + \frac{h}{2}\right), \text{ \&c.}$$

Now if we arrange these in tables in the usual order, as below, we

$\sin (x - 3h)$					
	$\Delta \sin (x - 3h)$				
$\sin (x - 2h)$		$\Delta^2 \sin (x - 3h)$			
	$\Delta \sin (x - 2h)$		$\Delta^3 \sin (x - 3h)$		
$\sin (x - h)$		$\Delta^2 \sin (x - 2h)$		$\Delta^4 \sin (x - 3h)$	
	$\Delta \sin (x - h)$		$\Delta^3 \sin (x - 2h)$		$\Delta^5 \sin (x - 3h)$
$\sin x$		$\Delta^2 \sin (x - h)$		$\Delta^4 \sin (x - 2h)$	
	$\Delta \sin x$		$\Delta^3 \sin (x - h)$		$\Delta^5 \sin (x - 2h)$
$\sin (x + h)$					

shall remark that $\sin x$, $\Delta^2 . \sin (x - h)$, $\Delta^4 \sin (x - 2h)$, &c. are in one horizontal line, and that $\Delta \sin x$, $\Delta^3 \sin (x - h)$, $\Delta^5 \sin (x - 2h)$, &c. are also in a horizontal line. Hence the numbers in each horizontal line form a geometrical progression, whose ratio is $- 4 \sin^2 \frac{h}{2}$. Knowing then $\sin x$ and $\sin (x + h)$, we can calculate all the differences as far as are necessary, and all our sines are then formed by addition and subtraction. If $x = 0$, we have but one series of differences to calculate.

(195.) By a slight alteration in the enunciation of this relation of the differences, we may avoid using any more numbers than are absolutely necessary. Since

$$\Delta^2 \sin x = - 4 \sin^2 \frac{h}{2} . \sin (x + h),$$

and $$\sin (x + h) = \sin x + \Delta \sin x,$$

therefore $\Delta^2 \sin x = - 4 \sin^2 \frac{h}{2} (\sin x + \Delta \sin x)$;

taking the $(n-2)^{\text{th}}$ difference of each side,

$$\Delta^n \sin x = - 4 \sin^2 \frac{h}{2} (\Delta^{n-2} \sin x + \Delta^{n-1} \sin x,)$$

a formula which gives any difference in terms of the two differences immediately preceding.

(196.) One important point we must not omit to notice, namely, the number of decimals to which these differences ought to be calculated. For this investigation we shall consider each of them as liable to the same error in the last figure used (it will never exceed half an unit, if we increase the last figure by 1 when the first rejected is equal to or greater than 5). Now it is useless to take one difference to so many decimals, that the error from it will be much less than that from any other; we shall, then, make them as nearly as possible equal. Suppose, now, there are n sines to be calculated by the differences, before our operations are verified by one of the previously calculated sines. The theory of finite differences gives us for

the $(n+1)^{\text{th}}$ sine, $\sin x + n \Delta \sin x + \frac{n \cdot (n-1)}{1 \cdot 2} \Delta^2 \cdot \sin (x - h)$

$$+ \frac{n \cdot (n-1) \cdot (n+1)}{1 \cdot 2 \cdot 3} \Delta^3 \cdot \sin (x-h)$$

$$+ \frac{n \cdot (n-1) \cdot (n+1) \cdot (n-2)}{1 \cdot 2 \cdot 3 \cdot 4} \Delta^4 \cdot \sin (x - 2h)$$

$$+ \frac{n \cdot (n-1) \cdot (n+1) \cdot (n-2) \cdot (n+2)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \Delta^5 \cdot \sin (x - 2h) + \&c.$$

The error of each difference will be multiplied, in the n^{th} sine, by the multiplier of that difference. If, then, $n = 59$, the first difference should be carried to 2 figures more than the sines, the second to 4, the third to 5, the fourth to 6, the fifth to 7, the sixth to 8, the seventh to 9, the eighth to 10, the ninth to 11, the tenth to 12. Or, if we make use of the differences calculated in (195) as the

$$(n+1)^{\text{th}} \text{ sine} = \sin x + n \Delta \sin x + \frac{n \cdot (n-1)}{1 \cdot 2} \Delta^2 \sin x$$

$$+ \frac{n \cdot (n-1) \cdot (n-2)}{1 \cdot 2 \cdot 3} \Delta^3 \sin x + \&c.,$$

we may in a similar manner find the number of decimal places to which each of these must be calculated. In adding any number to, or subtracting it from, any other number which has not so many decimals, we must not use the superabundant figures, but increase by 1 the first figure used, if the first of the superabundant figures be not less than 5. The sines with which we begin should be taken to 2 or 3 figures more than it is intended to preserve in the tables. In this way we can calculate with great accuracy, and without any unnecessary labour.

(197.) To interpolate for smaller divisions, as seconds, it is convenient to have a formula for finding the differences for the smaller divisions, by means of the differences for the larger ones. Suppose, now, the smaller divisions to be each $\frac{1}{p}^{th}$ of the large ones. Let Δ', Δ'', &c., be the 1st, 2d, &c., differences for minutes, and δ', δ'', &c., those for seconds. Then, by the common formula, we have

$$\sin x = \sin x$$

$$\sin(x + 1'') = \sin x + \frac{1}{p}\Delta' - \frac{1}{p}\left(1 - \frac{1}{p}\right)\frac{\Delta''}{1.2}$$

$$+ \frac{1}{p}\left(1 - \frac{1}{p}\right)\left(2 - \frac{1}{p}\right)\cdot\frac{\Delta'''}{1.2.3} - \text{\&c.}$$

$$= \sin x + \frac{1}{p}\Delta' - \left(\frac{1}{p} - \frac{1}{p^2}\right)\cdot\frac{\Delta''}{1.2}$$

$$+ \left(\frac{2}{p} - \frac{3}{p^2} + \frac{1}{p^3}\right)\cdot\frac{\Delta'''}{1.2.3}$$

$$- \left(\frac{6}{p} - \frac{11}{p^2} + \frac{6}{p^3} - \frac{1}{p^4}\right)\times\frac{\Delta''''}{1.2.3.4}$$

$$+ \left(\frac{24}{p} - \frac{50}{p^2} + \frac{35}{p^3} - \frac{10}{p^4} + \frac{1}{p^5}\right)\cdot\frac{\Delta'''''}{1.2.3.4.5}$$

and the sines of $(x + 2'')$, $(x + 3'')$, &c., will be found by putting $\frac{2}{p}$, $\frac{3}{p}$, &c., for $\frac{1}{p}$. Upon taking the differences of these successive values, it is clear that the numerator of $\frac{1}{p^m}$ in the n^{th} difference will be $\Delta^n . 0^m$ * multiplied by its factor in $\sin(x + 1'')$. Thus we find (going as far as the 5th differences)

* By $\Delta^n . 0^m$ is meant the first term of the n^{th} order of differences of the series 0^m, 1^m, 2^m, 3^m, &c.

$$\delta' = \frac{\Delta'}{p} - \frac{p-1}{2p^2}\,\Delta'' + \frac{(p-1).(2p-1)}{6p^3}\,\Delta'''$$
$$- \frac{(p-1).(2p-1).(3p-1)}{24p^4}\,\Delta''''$$
$$+ \frac{(p-1).(2p-1).(3p-1).(4p-1)}{120p^5}\,\Delta'''''.$$

$$\delta'' = \frac{1}{p^2}\,\Delta'' - \frac{p-1}{p^3}\,\Delta''' + \frac{(p-1).(11p-7)}{12p^4}\,\Delta'''$$
$$- \frac{(p-1).(2p-1).(5p-3)}{12p^5}\,\Delta'''''.$$

$$\delta''' = \frac{1}{p^3}\,\Delta''' - \frac{3.(p-1)}{2p^4}\,\Delta'''' + \frac{(p-1).(7p-5)}{4p^5}\,\Delta'''''.$$

$$\delta'''' = \frac{1}{p^4}\,\Delta'''' - \frac{2.(p-1)}{p^5}\,\Delta'''''.$$

$$\delta''''' = \frac{1}{p^5}\,\Delta'''''.$$

These expressions* are quite general; from the relation among the differences of natural sines mentioned in (194), it is not absolutely necessary to calculate more than the first of them; but even there it will be more convenient to use the formulæ.

* The demonstration in the text is the most simple, but the law may be found more generally in this manner. The problem is, from the given differences of the series, u_x, u_{x+p}, u_{x+2p}, &c. to find the differences of u_x, u_{x+1}, u_{x+2}, &c Let $\varphi(t)$ be the *Generating Function* of u_x; the generating functions of Δu_x, $\Delta^2 u_x$, &c. are $\left(\frac{1}{t^p}-1\right)\varphi(t)$, $\left(\frac{1}{t^p}-1\right)^2.\varphi(t)$ &c.; and those of δu_x, $\delta^2 u_x$, &c. are $\left(\frac{1}{t}-1\right)\varphi(t)$, $\left(\frac{1}{t}-1\right)^2.\varphi(t)$, &c. For $\delta^n.u_x$, then, we must express $\left(\frac{1}{t}-1\right)^n$ in powers of $\left(\frac{1}{t^p}-1\right)$. Let $\frac{1}{t^p}-1=z$; $\frac{1}{t}=(1+z)^{\frac{1}{p}}$; $\left(\frac{1}{t}-1\right)^n=\left(\overline{1+z}^{\frac{1}{p}}-1\right)^n$; let this $=\mathrm{A}\,z^n+\mathrm{B}\,z^{n+1}+$ &c.; therefore $\left(\frac{1}{t}-1\right)^n=\mathrm{A}.\left(\frac{1}{t^p}-1\right)^n+\mathrm{B}.\left(\frac{1}{t^p}-1\right)^{n+1}+$ &c., and $\left(\frac{1}{t}-1\right)^n\varphi(t)=\mathrm{A}.\left(\frac{1}{t^p}-1\right)^n\varphi(t)+\mathrm{B}.\left(\frac{1}{t^p}-1\right)^{n+1}.\varphi(t)+$ &c.; and taking the quantities of which these are the generating functions, $\delta^n.u_x=\mathrm{A}.\Delta_n.u_x+\mathrm{B}.\Delta^{n+1}.u_x+$ &c., where A, B, &c. are the coefficients of z^n, z^{n+1}, &c. in the expansion of $\left(\overline{1+z}^{\frac{1}{p}}-1\right)^n$.

(198.) The sines up to 60° being calculated, those above 60° will be found by simple addition, from the formula

$$\sin (60^\circ + \mathrm{A}) = \sin (60^\circ - \mathrm{A}) + \sin \mathrm{A}.$$

Thus the sines are found for the quadrant; and, consequently all the cosines are known

(199.) The tangents will be found by dividing the sines by the cosines. After 45° they may be found by the formula

$$\tan (45^\circ + \mathrm{A}) = \tan (45^\circ - \mathrm{A}) + 2 \tan 2\,\mathrm{A}.$$

(200.) The tangents may also be found independently in the following manner. If we expand every fractional term, except the first, of the first series in (157), and add together the coefficients of similar powers of x, and for x put $\frac{m}{n} \cdot \frac{\pi}{2}$ we have the following expression,

$$\tan \frac{m}{n} \cdot \frac{\pi}{2} = \frac{2\,m\,n}{n^2 - m^2} \times 0{,}6366197723675813$$

$$+ \frac{m}{n} \times 0{,}297556782059734 \qquad + \frac{m^3}{n^3} \times 0{,}018688650277330$$

$$+ \frac{m^5}{n^5} \times 0{,}001842475203510 \qquad + \frac{m^7}{n^7} \times 0{,}000197580071520$$

$$+ \frac{m^9}{n^9} \times 0{,}000021697737325 \qquad + \frac{m^{11}}{n^{11}} \times 0{,}000002401136991$$

$$+ \frac{m^{13}}{n^{13}} \times 0{,}000000266413303 \qquad + \frac{m^{15}}{n^{15}} \times 0{,}000000029586468$$

$$+ \frac{m^{17}}{n^{17}} \times 0{,}000000003286788 \qquad + \frac{m^{19}}{n^{19}} \times 0{,}000000000365175$$

$$+ \frac{m^{21}}{n^{21}} \times 0{,}000000000040754 \qquad + \frac{m^{23}}{n^{23}} \times 0{,}000000000004508$$

$$+ \frac{m^{25}}{n^{25}} \times 0{,}000000000000501 \qquad + \frac{m^{27}}{n^{27}} \times 0{,}000000000000056$$

$$+ \frac{m^{29}}{n^{29}} \times 0{,}000000000000006$$

Similarly, from the second series in (157), $\cot \frac{m}{n} \cdot \frac{\pi}{2} =$

$$\frac{n}{m} \times 0{,}6366197723675813 - \frac{4\,m\,n}{4n^2 - m^2} \times 0{,}3183098861837907$$

$$- \frac{m}{n} \times 0{,}205288889414508 \qquad - \frac{m^3}{n^3} \times 0{,}006551074788218$$

$$- \frac{m^5}{n^5} \times 0{,}000345029255397 \qquad - \frac{m^7}{n^7} \times 0{,}000020279106052$$

$$- \frac{m^9}{n^9} \times 0{,}000001236652718 \qquad - \frac{m^{11}}{n^{11}} \times 0{,}000000076495882$$

$$- \frac{m^{13}}{n^{13}} \times 0{,}000000004759738 \qquad - \frac{m^{15}}{n^{15}} \times 0{,}000000000296905$$

$$- \frac{m^{17}}{n^{17}} \times 0{,}000000000018541 \qquad - \frac{m^{19}}{n^{19}} \times 0{,}000000000001158$$

$$- \frac{m^{21}}{n^{21}} \times 0{,}000000000000072 \qquad - \frac{m^{23}}{n^{23}} \times 0{,}0000000000000035$$

The first fractional term in each expression is not expanded, as the series by that means are made to converge much more rapidly than if it were. It will never be necessary to take $\frac{m}{n}$ greater than $\frac{1}{2}$.

(201.) It is plain, however, that this process is too laborious to be applied to every one of the small divisions, and that it cannot with ease be extended farther than to every degree. But the calculation of differences of tangents admits of none of those simplifications which assisted us so much in forming tables of sines; we proceed, therefore, to give a method which applies to all cases whatever.

(202.) Let u be a function of $x = \phi\,(x)$; suppose x to receive the increments h, $2\,h$, &c.; then

$$\phi\,(x) = u.$$

$$\phi\,(x + h) = u + \frac{d\,u}{d\,x} \cdot \frac{h}{1} + \frac{d^2\,u}{d\,x^2} \cdot \frac{h^2}{1\,.\,2} + \frac{d^3\,u}{d\,x^3} \cdot \frac{h^3}{1\,.\,2\,.\,3} + \text{\&c.}$$

$$\phi\,(x + 2\,h) = u + \frac{d\,u}{d\,x} \cdot \frac{2\,.\,h}{1} + \frac{d^2\,u}{d\,x^2} \cdot \frac{2^2\,.\,h^2}{1\,.\,2} + \frac{d^3\,u}{d\,x^3} \cdot \frac{2^3\,.\,h^3}{1\,.\,2\,.\,3} + \text{\&c.}$$

Upon taking the differences it is evident, that (observing that $\Delta^n . 0^m$ is 0 when n is greater than m)

$$\Delta^n . u = \frac{\Delta^n . 0^n}{1\ 2 \ldots . n} \cdot \frac{d^n u}{d x^n} h^n + \frac{\Delta^n . 0^{n+1}}{1\ 2 \ldots . (n+1)} \cdot \frac{d^{n+1} u}{d x^{n+1}} h^{n+1} + \&c.$$

Now the numbers $\frac{\Delta^n . 0^n}{1 . 2 \ldots . n} h^n$, $\frac{\Delta^n . 0^{n+1}}{1\ 2 \ldots . (n+1)} h^{n+1}$, &c., can be conveniently calculated first (as they will be the same for every difference calculated thus) h being expressed supposing the radius $= 1$; and the formula for $\frac{d^n u}{d x^n}$ can also be found, and it will only be necessary at each calculation of differences to substitute numerical values in the expression for $\frac{d^n u}{d x^n}$. For the tangent $u = \tan x$, $\frac{d u}{d x} = \sec^2 x$, $\frac{d^2 u}{d x^2} = 2 \sec^2 x \quad \tan x$, &c.; and for intervals of a minute each, $h = 0{,}0002908882086657215 96$. The following table contains the values of $\frac{\Delta^n . 0^m}{1 . 2 \ldots . m}$ from $n = 1$ to $n = 12$, and from $m = 1$ to $m = 12$.

	$\frac{0^1}{1}$	$\frac{0^2}{1.2}$	$\frac{0^3}{1.2.3}$	$\frac{0^4}{1.2.3.4}$	$\frac{0^5}{1\ldots\ldots5}$	$\frac{0^6}{1\ldots\ldots6}$	$\frac{0^7}{1\ldots\ldots7}$	$\frac{0^8}{1\ldots\ldots8}$	$\frac{0^9}{1\ldots\ldots9}$	$\frac{0^{10}}{1\ldots\ldots10}$	$\frac{0^{11}}{1\ldots\ldots11}$	$\frac{0^{12}}{1\ldots\ldots12}$
Δ^1	1	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{24}$	$\frac{1}{120}$	$\frac{1}{720}$	$\frac{1}{5040}$	$\frac{1}{40320}$	$\frac{1}{362880}$	$\frac{1}{3628800}$	$\frac{1}{39916800}$	$\frac{1}{479001600}$
Δ^2		1	1	$\frac{7}{12}$	$\frac{1}{4}$	$\frac{31}{360}$	$\frac{1}{40}$	$\frac{127}{20160}$	$\frac{17}{12096}$	$\frac{73}{259200}$	$\frac{31}{604800}$	$\frac{2047}{239500800}$
Δ^3			1	$\frac{3}{2}$	$\frac{5}{4}$	$\frac{3}{4}$	$\frac{43}{120}$	$\frac{23}{160}$	$\frac{605}{12096}$	$\frac{311}{20160}$	$\frac{2591}{604800}$	$\frac{437}{403200}$
Δ^4				1	2	$\frac{13}{6}$	$\frac{5}{3}$	$\frac{81}{80}$	$\frac{37}{72}$	$\frac{6821}{30240}$	$\frac{265}{3024}$	$\frac{55591}{1814400}$
Δ^5					1	$\frac{5}{2}$	$\frac{10}{3}$	$\frac{25}{8}$	$\frac{331}{144}$	$\frac{45}{32}$	$\frac{2243}{3024}$	$\frac{1045}{3024}$
Δ^6						1	3	$\frac{19}{4}$	$\frac{21}{4}$	$\frac{1087}{240}$	$\frac{259}{80}$	$\frac{30083}{15120}$
Δ^7							1	$\frac{7}{2}$	$\frac{77}{12}$	$\frac{49}{6}$	$\frac{1939}{240}$	$\frac{4753}{720}$
Δ^8								1	4	$\frac{25}{3}$	12	$\frac{4819}{360}$
Δ^9									1	$\frac{9}{2}$	$\frac{21}{2}$	$\frac{135}{8}$
Δ^{10}										1	5	$\frac{155}{12}$
Δ^{11}											1	$\frac{11}{2}$
Δ^{12}												1

(203.) The same cautions as in (196) must be observed with regard to the number of decimals. And for the calculation for smaller divisions, as seconds, the formulæ of (197) must be used. Thus the table of tangents is completed.

(204.) The secants are calculated from the formula $\tan A + \cot A = 2 \operatorname{cosec} 2A$. This gives the cosecants or secants only for every second division; but the interpolation for every division will be sufficiently easy.

(205.) Thus then our tables of natural sines, tangents, and secants, is completed. The tables of their logarithms might be formed by taking from logarithmic tables the logarithms of these numbers; and many writers have considered this as being upon the whole the easiest way. As they may, however, be found independently, and therefore free from all errors of previous computations, and as the method appears to be the easiest, we shall give it here.

(206.) It has been seen (155) that

$$\sin x = x\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{4\pi^2}\right)\left(1-\frac{x^2}{9\pi^2}\right).\ \&c.,$$

and therefore $\log \sin x = \log x + \log\left(1-\frac{x^2}{\pi^2}\right)$

$$+\log\left(1-\frac{x^2}{4\pi^2}\right)+\log\left(1-\frac{x^2}{9\pi^2}\right)+\&c.$$

Expanding all the fractions but the first, and putting M for the modulus of common logarithms,

$$\log \sin x = \log x + \log\left(1-\frac{x^2}{\pi^2}\right)$$

$$-M\left\{\begin{array}{l} \frac{x^2}{4\pi^2}+\frac{1}{2}\cdot\frac{x^4}{16\pi^4}+\frac{1}{3}\cdot\frac{x^6}{64\pi^6}+\&c. \\ +\frac{x^2}{9\pi^2}+\frac{1}{2}\cdot\frac{x^4}{81\pi^4}+\frac{1}{3}\cdot\frac{x^6}{729\pi}+\&c. \\ +\&c. \end{array}\right\}$$

Adding the coefficients of similar powers of x, and putting $\frac{m}{n}\cdot\frac{\pi}{2}$ for x, we find the following series,

$$\log \sin \frac{m}{n}\cdot\frac{\pi}{2} = \log m + \log(2n-m)$$

$$+\log(2n+m) - 3\log n + 9{,}594059885702190$$

$$- \frac{m^2}{n^2} \times 0{,}070022826605902 \quad - \frac{m^4}{n^4} \times 0{,}001117266441662$$

$$- \frac{m^6}{n^6} \times 0{,}000039229146454 \quad - \frac{m^8}{n^8} \times 0{,}000001729270798$$

$$- \frac{m^{10}}{n^{10}} \times 0{,}000000084362986 \quad - \frac{m^{12}}{n^{12}} \times 0{,}000000004348716$$

$$- \frac{m^{14}}{n^{14}} \times 0{,}000000000231931 \quad - \frac{m^{16}}{n^{16}} \times 0{,}000000000012659$$

$$- \frac{m^{18}}{n^{18}} \times 0{,}000000000000703 \quad - \frac{m^{20}}{n^{20}} \times 0{,}000000000000040$$

$$- \frac{m^{22}}{n^{22}} \times 0{,}000000000000002.$$

And similarly $\log \cos \frac{m}{n} \frac{\pi}{2} = \log (n - m) + \log (n + m) - 2 \log n$

$$- \frac{m^2}{n^2} \times 0{,}101494859341893 \quad - \frac{m^4}{n^4} \times 0{,}003187294065451$$

$$- \frac{m^6}{n^6} \times 0{,}000209485800017 \quad - \frac{m^8}{n^8} \times 0{,}000016848348598$$

$$- \frac{m^{10}}{n^{10}} \times 0{,}000001480193987 \quad - \frac{m^{12}}{n^{12}} \times 0{,}000000136502272$$

$$- \frac{m^{14}}{n^{14}} \times 0{,}000000012981715 \quad - \frac{m^{16}}{n^{16}} \times 0{,}000000001261471$$

$$- \frac{m^{18}}{n^{18}} \times 0{,}000000000124567 \quad - \frac{m^{20}}{n^{20}} \times 0{,}000000000012456$$

$$- \frac{m^{22}}{n^{22}} \times 0{,}000000000001258 \quad - \frac{m^{24}}{n^{24}} \times 0{,}000000000000128$$

$$- \frac{m^{26}}{n^{26}} \times 0{,}000000000000013 \quad - \frac{m^{28}}{n^{28}} \times 0{,}000000000000001$$

(207.) If $\frac{m}{n}$ be small, the first terms in the last expression, which together $= \log \left(1 - \frac{m^2}{n^2}\right)$, may be expanded into the series

$$- 2\,\mathrm{M} \times \left\{ \frac{m^2}{2\,n^2 - m^2} + \frac{1}{3} \cdot \left(\frac{m^2}{2\,n^2 - m^2}\right)^3 + \&c. \right\},$$

where M = modulus = 0,434294481903252.

To make the logarithm positive, 10 must be added. This makes our operations entirely independent of logarithmic tables.

(208.) It is sufficient to find the log sines of the arcs between 45° and 90°, or the log cosines of arcs less than 45°. The remainder may be found thus,

$$\log \sin \mathrm{A} = 10 + \log \sin 2\,\mathrm{A} - \log \cos \mathrm{A} - \log 2$$
$$= \log \sin 2\,\mathrm{A} - \log \cos \mathrm{A} + 9{,}698970004336019.$$

By properly applying this theorem we may descend successively from log sin 45° to the log sines of all arcs less than 45°. By this method then the log sines and log cosines, and consequently the log tangents (since $\log \tan \mathrm{A} = 10 + \log \sin \mathrm{A} - \log \cos \mathrm{A}$) may be calculated for every degree.

(209.) If, however, the log sines be calculated independently for larger intervals, as for every 10°, the differences for every degree may be thus found,

$$\log \sin (x+h) - \log \sin x = \log \frac{\sin (x+h)}{\sin x}$$
$$= 2\,\mathrm{M} \left\{ \frac{\sin (x+h) - \sin x}{\sin (x+h) + \sin x} + \frac{1}{3} \cdot \left(\frac{\sin (x+h) - \sin x}{\sin (x+h) + \sin x} \right)^3 + \&c. \right\}$$

a series which converges rapidly. Or, when one first difference is thus found, the second differences may be calculated by this series,

$$\Delta^2 \log \sin x = \log \frac{\sin x \sin (x + 2\,h)}{\sin^2 (x+h)}$$
$$= -\,2\,\mathrm{M} \left\{ \frac{\sin^2 (x+h) - \sin x \,.\, \sin (x+2\,h)}{\sin^2 (x+h) + \sin x \,.\, \sin (x+2\,h)} \right.$$
$$\left. + \frac{1}{3} \left(\frac{\sin^2 (x+h) - \sin x \,.\, \sin (x+2\,h)}{\sin^2 (x+h) + \sin x \,.\, \sin (x+2\,h)} \right)^3 + \&c. \right\};$$

which, since

$$\frac{\sin^2 (x+h) - \sin x \,.\, \sin (x+2\,h)}{\sin^2 (x+h) + \sin x \,.\, \sin (x+2\,h)} = \frac{\sin^2 h}{\cos^2 h + \cos (2\,x + h)},$$

converges much more rapidly.

(210.) Before proceeding farther, it will be proper to verify the numbers already calculated; and here the formula of (159) will be found very useful. For taking the logarithms of both sides of that equation, $\log \sin n\,\beta = (n-1) \times 0{,}3010299956639812 + \log \sin \beta$

$$+ \log \sin \left(\beta + \frac{\pi}{n} \right) + \log \sin \left(\beta + \frac{3\,\pi}{n} \right) + \&c. \text{ to } n \text{ terms},$$

where n and β may be taken at pleasure.

(211.) It is then best to fill them up by differences; and the differences may be calculated in the same manner as in (202.)

Here $$\frac{d\,u}{d\,x} = \mathrm{M} \cot x\,;\ \frac{d^2\,u}{d\,x^2} = -\,\mathrm{M}\,(1 + \cot^2 x)\,;$$

$$\frac{d^3\,u}{d\,x^3} = 2\,\mathrm{M} \cot x\,(1 + \cot^2 x), \text{ \&c.}$$

The calculation of the differences is rather tedious, but the tables are formed then with great ease, and the certainty that any error will be discovered at the next place of verification makes this method superior to any other.

(212.) For the smaller divisions, the differences will be found from these differences by the formulæ in (197). Thus our tables of logarithmic sines, cosines, and tangents will be completed.

(213.) It is unnecessary to examine by any formula of verification the accuracy of the numbers for the small divisions of the arc. It is scarcely possible to have a better verification, than the agreement of the last of a series of numbers computed by differences with one which has previously been calculated by an independent process.

(214.) In (165) we have alluded to tables of the logarithms of $\frac{\sin x}{x}$ for a few degrees. These are calculated very easily from the series $\log \frac{\sin x}{x} = -\,\mathrm{M}\left\{\frac{x^2}{6} + \frac{x^4}{3^2\,.\,4\,.\,5} + \frac{x^6}{3^4\,.\,5\,.\,7} + \text{\&c.}\right\}$

When $x = 5°$, the third term has no significant figure in the first ten decimal places. For tables to 10 decimals the first term is sufficient up to 1°, and the first two terms to 5°. For tables to 7 decimals, the first term is sufficient, as the second term produces 1 in the last place when $x = 5°$. This, therefore, is easily calculated by second differences. If the $\log \frac{\tan x}{x}$ be required, since $\frac{\tan x}{x} = \frac{\sin x}{x} \cdot \frac{1}{\cos x}$, we have

$$\log \frac{\tan x}{x} = \log \frac{\sin x}{x} + \text{ar. comp. } \log \cos x\,;$$

or it can be calculated in the same way.

(215.) Since $\sec x = \frac{1}{\cos x}$, its logarithm will be immediately found. And since $\operatorname{versin} x = 1 - \cos x = 2 \sin^2 \frac{x}{2}$, the natural and logarithmic versed sines are found. They are seldom inserted in tables, except in those employed in nautical astronomy.

(216.) The principal tables commonly in use are the following:— Sherwin's, containing, besides the logarithms of numbers, sines, cosines, tangents, &c., natural and logarithmic, for every minute, to 7 decimals; Hutton's, containing the same, with an interesting and valuable introduction; Gardiner's, with log. sines, &c., for every 10 seconds to 7 decimals; Taylor's, with log. sines, &c., to 7 decimals for every second. Of these the most common is Hutton's. Many smaller collections of tables are in use. Of the foreign tables, the best are Vega's, containing the logarithms of numbers and log. sines, &c., for every $10''$ to 10 decimals; Callet's logarithms of numbers, log. sines, &c., for every $10''$ to 7 decimals, with some tables for the decimal division of the circle. This is a very convenient and useful collection. An abridged form of the *Tables du Cadastre*, revised by Delambre, has, we believe, been edited by Borda; and must form a useful collection for the decimal division.

(217.) Trigonometrical tables have generally sines, cosines, tangents, cotangents, &c., up to $45°$; the cotangent of an arc being the tangent of its complement, &c. What is gained by this arrangement, except perhaps in the use of subsidiary angles, it is not easy to say; and in taking out the sine, &c., of an arc greater than $45°$, or greater than $90°$, there is frequently some confusion. We should prefer the more natural arrangement of sines, tangents, &c., up to $90°$; these read in the reverse order (as shown by the figures and titles at the bottom of the page) would give the cosines, cotangents, &c.

GLASGOW: PRINTED BY BELL AND BAIN, 15 ST. ENOCH SQUARE.

IMPORTANT WORKS

January, 1855.

In Science and General Literature,

PUBLISHED BY

RICHARD GRIFFIN AND COMPANY,

PUBLISHERS TO THE UNIVERSITY OF GLASGOW,

VIA · VERITAS · VITA

5 Warwick Square, PATERNOSTER ROW, LONDON.

Prince of Wales Buildings, BUCHANAN-STREET, GLASGOW.

Classified Index.

ABBOTT.—MORAL AND EDUCATIONAL WORKS:
Comprising the Young Christian, Teacher, Way to do Good, &c. 8vo. 5s. cloth.

AIRY.—CONTRIBUTIONS TO THE ENCYCLOPÆDIA METROPOLITANA:
viz. Trigonometry, Figure of the Earth, and Tides and Waves. One Volume, 4to, 10s. 6d. cloth.

AIRY.—TRIGONOMETRY.
A TREATISE ON TRIGONOMETRY. By GEORGE B. AIRY, F.R.S., Astronomer Royal. New edition with Questions. Crown 8vo, cloth.

ANCIENT HISTORY,
Comprising the Political, Social, and Literary Histories of ROME, GREECE, EGYPT, ASSYRIA, PERSIA, &c. &c. By ARNOLD, TALFOURD, RAMSAY, EADIE, RENOUARD, WHEWELL, &c., &c, 10 vols., crown 8vo, cloth, gilt, 4*l*. 4s.

ANECDOTES—LITERARY AND SCIENTIFIC.
CYCLOPÆDIA OF LITERARY AND SCIENTIFIC ANECDOTE. Edited by WILLIAM KEDDIE, Esq., Secretary to the Glasgow Philosophical Society. Crown 8vo, 5s. cloth.

ANECDOTES—RELIGIOUS.
CYCLOPÆDIA OF RELIGIOUS ANECDOTE; A vast Collection of Facts, Narratives, and Examples, with Copious Index, and Introductory Essay by Rev. Dr. CHEEVER. To which is added a Complete Series of Scriptural Texts illustrated by the Anecdotes. 4th edition. Crown 8vo, 5s. cloth.

ASTRONOMY.
THE ENCYCLOPÆDIA OF ASTRONOMY.—Plane Astronomy, (Professor BARLOW,) Nautical Astronomy, (Captain KATER,) Physical Astronomy, (Sir J. HERSCHEL,) Figure of the Earth, (Professor AIRY,) Tides and Waves, (Professor AIRY.) With 21 Plates, 4to, 14s. cloth.

BALFOUR.—BOTANY;
An Introduction to the Study of the Structure, Physiology, and Classification of Plants. By JOHN H. BALFOUR, M.D., F.R.S.E., Professor of Botany in the University of Edinburgh. Numerous Illustrations, Third Edition, crown 8vo.

BIOGRAPHY, CYCLOPÆDIA OF;
Containing a Complete Series of Original Memoirs of the most Remarkable Individuals of all Times and Nations. By ALISON, BREWSTER, BURTON, CREASY, EADIE, KNIGHT, NICHOL, SPALDING, WORNUM, and other Contributors. With 150 Illustrations of Birth-places, &c. &c. Post 8vo, 12s. 6d. cloth.

BOOK OF NATURE:
THE BOOK OF NATURE; a COMPREHENSIVE INTRODUCTION to the NATURAL and PHYSICAL SCIENCES. From the German of Professor SCHOEDLER, with numerous Additions, by HENRY MEDLOCK, F.C.S. Second edition, corrected and enlarged. With Copious Index, and 500 Engravings, post 8vo, cloth, 10s. 6d.

BOOK OF TRADES;
Or, Circle of the Useful Arts. Illustrated by Engravings and Woodcuts, Eleventh Edition, 16mo, 4s. 6d. cloth.

BRITISH ELOQUENCE:
THE LITERARY, POLITICAL, and SACRED ORATORY of the NINETEENTH CENTURY, in a series of foolcap 8vo volumes. 3s. 6d. each, cloth.

Literary Oratory.—A Series of Literary Addresses, by BULWER LYTTON, BROUGHAM, MACAULAY, MASSON, STEPHEN, and others. Foolscap 8vo. 3s. 6d. cl.

Political Oratory.—The Great Speeches of the Great Statesmen from the Close of the War to the Passing of the Reform Bill. Foolscap 8vo, 3s. 6d., cloth.

BROUGHAM.—WORKS.
WORKS of the Right Honourable LORD BROUGHAM, comprising his Sketches of Men of Letters, Philosophers and Statesmen, Literary and Scientific Discourses, Contributions to Reviews, &c., &c. Now first Collected under the direct care and superintendence of his Lordship, in a series of post 8vo volumes, each 5s. cloth.

BRYCE.—UNIVERSAL GAZETTEER.
A CYCLOPÆDIA OF POLITICAL AND PHYSICAL GEOGRAPHY, with Pronunciation, and numerous Illustrations. By JAMES BRYCE, A.M., F.G.S. Post 8vo.

CAMPBELL.—BRITISH NAVAL HISTORY.
Lives of the British Admirals and Naval History of Great Britain from the Days of Cæsar to the Present Time. Fifth Edition, with numerous engravings, foolscap 8vo, 3s. 6d. cloth, or 4s. extra gilt.

ECCLESIASTICAL HISTORY.

FIRST DIVISION.—HISTORY OF THE CHRISTIAN CHURCH IN THE FIRST CENTURY. By Right Rev. SAMUEL HINDS, D.D., Bishop of Norwich, and Rev. J. H. NEWMAN, B.D. Crown 8vo, 6s. cloth.

SECOND DIVISION.—CHRISTIANITY IN THE MIDDLE AGES.

Part I.—THE CHRISTIAN CHURCH IN THE SECOND AND THIRD CENTURIES, by JAMES A. JEREMIE, D.D., Regius Professor of Divinity in the University of Cambridge. 4s., cloth.

Part II.—THE CHRISTIAN CHURCH FROM THE FOURTH TO THE TWELFTH CENTURY. By Rev. J. M. GUILDING, Rev. J. B. S. CARWITHEN, B.D., &c.

THIRD DIVISION.—THE RISE, ACCOMPLISHMENT, & CONSEQUENCES OF THE REFORMATION; being the History of the Church from the Thirteenth Century to the present time. By Right Reverend Dr. HAMPDEN, Bishop of Hereford, Rev. J. E. RIDDLE, M.A., &c. &c.

ENCYCLOPÆDIA METROPOLITANA, NEW AND REVISED

EDITION. The Second Edition of the Encyclopædia Metropolitana will be handsomely printed in a Series of Cabinet Volumes, each containing a Complete Treatise, in crown 8vo. The work will be thoroughly revised, many new Treatises added, the Articles all provided with comprehensive Indexes, or Analytical Tables of Contents, and abundantly illustrated by Maps, Woodcuts, and Engravings.

Vol. 1.—METHOD.	S. T. COLERIDGE. 2s.
2.—UNIVERSAL GRAMMAR.	Sir JOHN STODDART. 5s.
3.—LOGIC.	ARCHBISHOP WHATELY. 3s.
4.—RHETORIC.	ARCHBISHOP WHATELY. 3s. 6d.
5.—EARLY CHRISTIANITY.	BISHOP HINDS. 6s.
6.—POLITICAL ECONOMY.	NASSAU W. SENIOR. 4s.
7.—HISTORY OF THE JEWS.	ARCHDEACON HALE. 2s. 6d.
8.—SACRED HISTORY AND BIOGRAPHY.	DR. COX. 6s.
9.—GREEK LITERATURE.	Sir T. N. TALFOURD, &c. 7s. 6d.
10.—ANCIENT PHILOSOPHY.	REV. F. D. MAURICE. 5s.
11.—UNIVERSAL HISTORY.	Sir JOHN STODDART. 5s.
12.—ROMAN ANTIQUITIES.	PROFESSOR RAMSAY. 8s. 6d.
13.—BOTANY.	PROFESSOR BALFOUR.
14.—ELECTRO-METALLURGY.	JAMES NAPIER. 3s. 6d.
15.—EARLY HISTORY OF GREECE.	Sir T. N. TALFOURD, &c. 9s.
16.—PHOTOGRAPHY.	ROBERT HUNT. 6s.
17.—VETERINARY ART.	W. C. SPOONER. 3s.
18.—EARLY ORIENTAL HISTORY.	PROFESSOR EADIE. 8s.
19.—HISTORY OF THE ROMAN REPUBLIC.	DR. ARNOLD, &c. 8s. 6d.
20.—BIBLICAL ANTIQUITIES.	DR. COX. 7s. 6d.
21.—METALLURGY.	J. A. PHILLIPS. 12s. 6d.
22.—THE CHURCH IN THE SECOND AND THIRD CENTURIES.	PROFESSOR JEREMIE. 4s.
23.—HISTORY OF GREECE & MACEDONIA.	DEAN LYALL, &c. 8s.
24.—ROMAN LITERATURE.	DR. ARNOLD, &c. 10s. 6d.
25.—HISTORY OF THE ROMAN EMPIRE.	DR. ARNOLD, &c. 10s. 6d.
26.—DECLINE AND FALL OF ROME.	BISHOP RUSSEL, &c. 10s. 6d.
27.—GREEK AND ROMAN PHILOSOPHY.	BISHOP OF LONDON, &c. 6s.
28.—PHILOSOPHY OF THE FIRST SIX CENTURIES.	REV. F. D. MAURICE. 3s. 6d.
29.—HISTORY OF THE OTTOMAN EMPIRE.	Col. PROCTER, &c. 7s. 6d.
30.—TRIGONOMETRY.	PROFESSOR AIRY.
31.—OCCULT SCIENCES.	REV. E. SMEDLEY, &c.
32.—GEOLOGY.	PROFESSOR PHILLIPS.

GREEK AND ROMAN PHILOSOPHY AND SCIENCE.

HISTORY OF GREEK AND ROMAN PHILOSOPHY AND SCIENCE. By the Rev. DR. BLOMFIELD, WILLIAM LOWNDES, Esq., M.A., Q.C., Rev. J. W. BLAKESLEY, M.A., ANDREW FINDLATER, A.M., Rev. Professor JEREMIE, Rev. Professor WHEWELL, and other Contributors. Crown 8vo, 6s. cloth.

GREECE.—HISTORY OF.

HISTORY OF GREECE from the Earliest Period to the absorption of Greece in the Roman Empire. By E. POCOCKE, Esq., Rev. J. B. OTTLEY, M.A., Sir THOMAS N. TALFOURD, D.C.L., the late J. T. RUTT, Esq., WILLIAM ROE LYALL, D.D., Rev. J. H. B. MOUNTAIN, D.D., Rev. G. C. RENOUARD, B.D., and others. Illustrated by numerous Wood Engravings. 2 vols. crown 8vo, 17s. cloth.

1. EARLY HISTORY OF GREECE. 150 Engravings, 9s. cloth.
2. HISTORY OF GREECE AND MACEDONIA. 100 Engravings, 8s.

GREEK LITERATURE.

A HISTORY OF GREEK LITERATURE. By Sir T. N. TALFOURD, Right Rev. Dr. BLOMFIELD, Bishop of London, Rev. H. THOMPSON, M.A., and other Contributors. With Chronological Tables and Index. Crown 8vo, 7s. 6d. cloth.

GRIFFIN'S PORTABLE CYCLOPÆDIAS.

A series of Alphabetical Reference Books adapted to the wants of the present age; beautifully printed and embellished. Post 8vo, cloth.

Vol. 1.—BIBLICAL KNOWLEDGE.	PROFESSOR EADIE. 7s. 6d.
2.—CHEMISTRY AND MINERALOGY.	DR. THOMSON. 12s. 6d.
3.—UNIVERSAL BIOGRAPHY.	ALISON, BREWSTER, &c. 12s. 6d
4.—UNIVERSAL HISTORY.	M'BURNEY and NEIL. 10s. 6d.
5.—PHYSICAL SCIENCES.	PROFESSOR NICHOL.
6.—UNIVERSAL GAZETTEER.	JAMES BRYCE, F.G.S,

HALE.—HISTORY OF THE JEWS,

From the Time of Alexander the Great to the Destruction of Jerusalem by Titus. By Archdeacon HALE, Master of the Charter House. Crown 8vo, 2s. 6d. cloth.

HERSCHEL.—CONTRIBUTIONS TO THE ENCYCLOPÆDIA

METROPOLITANA: viz. Physical Astronomy, Light, and Sound. 4to, cloth, 15s.

HINDS.—EARLY CHRISTIANITY.

THE RISE AND EARLY PROGRESS OF CHRISTIANITY. By Right Rev. Dr. HINDS, Bishop of Norwich. New Edition, Revised, crown 8vo, 5s. cloth.

HORATII OPERA.

THE COMPLETE WORKS OF HORACE, from the Text of ORELLIUS, with 250 Illustrations from the most authentic sources, and Introductory Dissertation on the Life and Poetry of Horace, by the Rev. HENRY THOMPSON, M.A., late Scholar of Cambridge, Vicar of Chard. Crown 8vo, 7s. 6d. antique binding.

HUNT.—PHOTOGRAPHY.

Embracing the Daguerreotype, Calotype, and all the published Photographic processes. By ROBERT HUNT, F.R.S. Fourth edition, enlarged. Numerous Engravings, crown 8vo, 6s. cloth.

JAMIESON.—RELIGIOUS BIOGRAPHY.

POPULAR CYCLOPÆDIA OF MODERN RELIGIOUS BIOGRAPHY. By the Rev. ROBERT JAMIESON, D.D. Second Edition, crown 8vo, 5s., cloth.

JEREMIE.—CHURCH HISTORY.

HISTORY OF THE CHRISTIAN CHURCH IN THE SECOND AND THIRD CENTURIES. By JAMES AMIRAUX JEREMIE, D.D., Regius Professor of Divinity, Cambridge. Crown 8vo, 4s. cloth.

KERR.—MANUFACTURE OF SUGAR;

A Practical Treatise on the Cultivation of the Sugar Cane and the Manufacture of Sugar. By THOMAS KERR, of Barbadoes. Crown 8vo, 5s. cloth.

KINGSLEY.—NATIONAL SERMONS.

By the Rev. CHARLES KINGSLEY, Canon of Middleham, and Rector of Eversley, author of Yeast, Alton Locke, &c. First and Second Series, fcap. 8vo, each 5s. cl.

MANUFACTURES AND MACHINERY.

THE ENCYCLOPÆDIA OF ARTS, MANUFACTURES, AND MACHINERY. By Professor BARLOW. With an Introductory Dissertation, by Professor BABBAGE. New edition, with 87 Pages of Engravings. 4to, 42s. half russia.

MATHEMATICS.

THE ENCYCLOPÆDIA OF PURE MATHEMATICS, comprehending a complete Course of Mathematical Science. By Professors AIRY, BARLOW, DE MORGAN, HALL, HAMILTON, LEVY, MOSELEY, Dr. LARDNER, F.R.S., and Rev. Dr. PEACOCK, Dean of Ely. With 17 Engravings, £1 1s. cloth.

MAURICE.—MORAL AND METAPHYSICAL PHILOSOPHY:

ANCIENT PHILOSOPHY, comprising the Hebrew, Egyptian, Hindoo, Chinese. Persian, Grecian, Roman and Alexandrian Systems of Philosophy. By Rev. F. D, MAURICE, M.A., Chaplain to Lincoln's Inn. Third Edition, cr. 8vo, 5s. cloth.

PHILOSOPHY OF THE FIRST SIX CENTURIES. Crown 8vo, 3s. 6d. cloth.

PHILOSOPHY OF THE MIDDLE AGES. Crown 8vo. *In the Press.*

MODERN PHILOSOPHY. Crown 8vo. *In the Press.*

MECHANICAL PHILOSOPHY.

THE ENCYCLOPÆDIA OF MECHANICAL PHILOSOPHY.—Mechanics, Pneumatics, Sound, Light, illustrated by 78 Plates. By Professor BARLOW and Sir JOHN HERSCHEL. 4to, £1 11s. 6d. cloth.

MEDICAL SCIENCE.

THE ENCYCLOPÆDIA OF THE MEDICAL SCIENCES. By J. F. SOUTH, Esq., F.L.S., F. LE GROS CLARK, Esq., W. BOWMAN, Esq., F.R.S., G. JOHNSON, Esq., M.D., R. WILLIAMS, Esq., M.D., and W. C. SPOONER, Esq. With 18 Engravings, 14s. cloth.

MENTAL SCIENCE.

THE ENCYCLOPÆDIA OF MENTAL PHILOSOPHY. By S. T. COLERIDGE, Sir J. STODDART, LL.D., Archbishop WHATELY, R. JEBB, Esq., A. POLSON, Esq., Rev. F. D. MAURICE, Profs. GRAVES and CORRIE, &c. 4to, 14s. cl.

MENTAL SCIENCE.

SAMUEL TAYLOR COLERIDGE ON METHOD; ARCHBISHOP WHATELY'S TREATISES ON LOGIC and RHETORIC. Crown 8vo, 5s. cloth.

MITCHISON.—SCOTTISH SONG.

A HAND-BOOK OF THE SONGS OF SCOTLAND, containing the best Songs of Burns, &c. Set to Music. With Notes, and Life of Wilson. By WILLIAM MITCHISON. Crown 8vo, 2s. 6d. cloth.

MORRISON.—BOOK-KEEPING;

A Complete System of Practical Book-keeping by Single Entry, Double Entry, and a New Method. By C. MORRISON. Eighth edition, 8vo., 8s. half-bound.

NAPIER.—ELECTRO-METALLURGY;

Containing an Account of the most Improved Methods of Depositing Copper, Silver, Gold, and other metals, with numerous Illustrations. By JAMES NAPIER, F.C.S. Second edition, revised and enlarged. Crown 8vo, 3s. 6d. cloth.

NAPIER.—DYEING.

A MANUAL OF DYEING, Practical and Theoretical. By JAMES NAPIER, F.C.S. With Engravings. Post 8vo, 7s. 6d. cloth.

NAPIER.—DYEING RECEIPTS.

A MANUAL OF DYEING RECEIPTS, with Specimens of Cloth. By JAMES NAPIER, F.C.S. Post 8vo, cloth.

NATURAL HISTORY.

THE ENCYCLOPÆDIA OF NATURAL HISTORY. By THOMAS EDWARDS, Esq., F.L.S., GEORGE DON, Esq., F.L.S., J. H. BROOKE, Esq., J. F. SOUTH, Esq., and Profs. PHILLIPS and DAUBENY. With 131 quarto Plates, 4to, £1 15s. cloth.

NATURAL SCIENCES.

INTRODUCTORY MANUAL OF THE NATURAL SCIENCES, comprising Geology, Botany, and Zoology. By Professors SCHOEDLER and MEDLOCK. Numerous Illustrations, crown 8vo, 5s. cloth.

NICHOL.—PHYSICAL SCIENCES.

A CYCLOPÆDIA OF THE PHYSICAL SCIENCES, comprising Acoustics—Astronomy—Dynamics — Electricity — Heat — Hydrodynamics — Magnetism — Philosophy of Mathematics—Meteorology—Optics—Pneumatics—Statics. By J. P. NICHOL, LL.D., Professor of Astronomy in the University of Glasgow. With Numerous Illustrations. Post 8vo.

OTTOMAN EMPIRE.—HISTORY OF.

HISTORY OF THE OTTOMAN EMPIRE, including a Sketch of the Greek Empire and the Crusades. By Col. PROCTER, Rev. J. E. RIDDLE, and J. M'CONECHY. Cr. 8vo, 7s. 6d.

PHILLIPS.—METALLURGY;

A MANUAL OF METALLURGY: being an Account of Assaying, Mining, Smelting, &c., by J. A. PHILLIPS, F.C.S. Second Edition, revised and enlarged, with above 200 Engravings. Crown 8vo, 12s. 6d. cloth.

PHILLIPS.—GOLD MINING;

A POPULAR TREATISE on GOLD-MINING, and ASSAYING. By J. A. PHILLIPS, F.C.S., F.G.S. With Illustrations, second edit., fols. 8vo, 2s. 6d. cloth.

PHILLIPS.—GEOLOGY.

A MANUAL OF GEOLOGY, Practical and Theoretical, with Numerous Illustrations. By JOHN PHILLIPS, M.A., F.R.S., F.G.S., Deputy Reader of Geology in the University of Oxford. Crown 8vo.

PHYSICAL SCIENCES.

INTRODUCTORY MANUAL OF THE PHYSICAL SCIENCES, comprising Natural Philosophy, Astronomy, and Chemistry. By Professors SCHOEDLER and MEDLOCK. Numerous Illustrations, crown 8vo, cloth, 5s.

POCOCKE.—INDIA IN GREECE,

Or, TRUTH IN MYTHOLOGY, containing an Account of the Sources of the Hellenic Race, the Colonisation of Egypt and Palestine from India, the Wars of the Grand Lama, and the Bud'histic Propaganda in Greece. By E. POCOCKE, Esq. Illustrated by Numerous Maps of India and Greece, post 8vo, 12s. cloth.

POLSON.—PRINCIPLES OF THE LAW OF NATIONS,

With Practical Notes on the Law of Blockade, and on Contraband in War. By ARCHER POLSON, Esq., Lincoln's Inn. To which is added DIPLOMACY. By THOMAS HARTWELL HORNE, B.D. Crown 8vo, 3s. 6d. cloth.

POPE.—POETICAL WORKS.

COMPLETE POETICAL WORKS and TRANSLATIONS of ALEXANDER POPE. Portrait and Vignette. New Edition, 8vo, 5s. cloth.

POPULAR LIBRARY;

A Miscellany of Amusement and Instruction; numerous Engravings, 12mo, 3s. 6d. cl.

PROCTER—CRUSADES.

HISTORY OF THE CRUSADES; their Rise, Progress, and Results. By Colonel PROCTER. With Illustrations after Gilbert, &c. Crown 8vo, 5s. cloth.

RAMSAY.—ROMAN ANTIQUITIES;

A MANUAL OF ROMAN ANTIQUITIES. By WILLIAM RAMSAY, M.A., Professor of Humanity in the University of Glasgow. With Map, numerous Engravings, and very copious Index. Third edition, crown 8vo, 8s. 6d. cloth.

REID.—INTELLECTUAL POWERS.

ESSAYS ON THE INTELLECTUAL POWERS OF MAN. By THOMAS REID, D.D. New edition, crown 8vo, 5s.

RELIGIOUS DENOMINATIONS.

A CYCLOPÆDIA OF RELIGIOUS DENOMINATIONS. An Authentic Account of the various Religions prevailing throughout the World, written by Members of the respective Bodies. Second edition, crown 8vo, 5s. cloth.

ROME—HISTORY OF.

HISTORY OF ROME, from the Foundation of the City of Rome to the Extinction of the Western Empire. By Rev. THOMAS ARNOLD, D.D., the Rev. Professor JEREMIE, D.D., SIR THOMAS NOON TALFOURD, D.C.L., and others. Illustrated by numerous Engravings. Three Volumes, crown 8vo, cl. extra, £1 9s. 6d.

1. THE ROMAN REPUBLIC, 8s. 6d. cloth.
2. THE ROMAN EMPIRE, 10s. 6d. cloth.
3. THE DECLINE AND FALL OF THE ROMAN POWER, 10s. 6d. cloth.

ROMAN LITERATURE.

A HISTORY OF ROMAN LITERATURE. By Rev. THOMAS ARNOLD, D.D., Rev. HENRY THOMPSON, M.A., Rev. J. H. NEWMAN, B.D., the Rev. J. M. NEALE, M.A., and other Contributors. Crown 8vo, 10s. 6d. cloth.

SCHOEDLER AND MEDLOCK'S SCIENTIFIC TREATISES,

FOR SCHOOLS. New Editions, post 8vo, cloth.

ASTRONOMY. With 51 Engravings. 1s. 6d.
BOTANY. With 121 Engravings. 1s. 6d.
CHEMISTRY. Second Edition. With 50 Engravings. 2s.
MINERALOGY and GEOLOGY. With 128 Engravings. 2s. 6d.
NATURAL PHILOSOPHY. With 150 Engravings. 2s. 6d.
ZOOLOGY and PHYSIOLOGY. With 84 Engravings. 2s.

SCRIPTURE READINGS—JAMIESON.

THE BIBLE FAMILIARLY EXPLAINED TO THE YOUNG. Edited by the Rev. Dr. JAMIESON. With Illustrations, foolscap 8vo, 3s. 6d . cloth.

SENIOR.—POLITICAL ECONOMY;

The Science which Treats of the Nature, Production, and Distribution of Wealth. By NASSAU W. SENIOR, M.A. Third edition, crown 8vo, 4s. cloth.

SMEDLEY.—OCCULT SCIENCES.

A COMPENDIUM OF THE OCCULT SCIENCES, comprising Supernatural Beings—Exercise of Occult Powers—Localities—Monsters—Divination—Charms—Wonderful Plants and Remedies — Miscellaneous Superstitions, &c. &c., by Rev. E. SMEDLEY, M.A., &c. Crown 8vo.

SPOONER.—VETERINARY ART.

A Practical Treatise on the Diseases of the Horse. By W. C. SPOONER, Esq. With 50 Engravings. Crown 8vo, 3s. cloth.

STODDART.—INTRODUCTION TO HISTORY;

Two Dissertations—FIRST: On the Uses of History as a Study. SECOND: On the Separation of the Early Facts of History from Fable. By Sir JOHN STODDART, LL.D. Crown 8vo, 5s. cloth.

STODDART.—UNIVERSAL GRAMMAR,

Or, the Pure Science of Language. By Sir JOHN STODDART, LL.D. Third edition, revised, crown 8vo, 5s. cloth.

THOMSON.—CHEMISTRY.

A CYCLOPÆDIA OF CHEMISTRY, PRACTICAL AND THEORETICAL, with its Applications to Mineralogy, Physiology, and the Arts. By R. D. THOMSON, M.D., F.R.S., F.C.S., Professor of Chemistry at St. Thomas's Hospital College, London. With numerous Illustrations, post 8vo, 12s. 6d. cloth.

UNIVERSAL HISTORY.

A CYCLOPÆDIA OF HISTORY. A COMPLETE SERIES OF CHRONOLOGICAL TABLES from the Beginning of the World to the Present Time, on a New Plan, comprehending an arrangement according to Dates, and an Alphabetical arrangement. Post 8vo. 10s. 6d., cloth.

USEFUL ARTS.

THE ENCYCLOPÆDIA OF THE USEFUL ARTS.—Agriculture, Horticulture, Commerce, Political Economy, Carpentry, Fortification, and Naval Architecture. By DON, LOW, SENIOR, &c. With 19 plates, 4to, 14s. cloth.

VIRGILII OPERA.

THE COMPLETE WORKS OF VIRGIL, from the text of Heyne and Wagner, with 300 Illustrations, from the most authentic sources, and Introductory Dissertation on the Life and Poetry of Virgil. By Rev. HENRY THOMPSON, M.A., late scholar of Cambridge, Vicar of Chard, &c. Crown 8vo, 8s. 6d., antique binding.

WHATELY.—LOGIC.

Original Edition, complete, forming part of the Encyclopædia Metropolitana. With Synopsis and Index. Crown 8vo, 2s. sewed, or 3s. cloth.

WHATELY.—RHETORIC.

Original Edition, complete, forming part of the Encyclopædia Metropolitana. With Synopsis and Index. Crown 8vo, 2s. 6d. sewed, or 3s. 6d. cloth.

WHITE.—THE NATURAL HISTORY OF SELBORNE.

THE NATURAL HISTORY OF SELBORNE. By Rev. GILBERT WHITE, A.M. New edition, greatly enlarged by Captain THOMAS BROWN, F.L.S., &c. With coloured Illustrations, foolscap 8vo, cloth, 3s. 6d., or richly gilt, 4s.

WITTICH.—PHYSICAL GEOGRAPHY.

CURIOSITIES OF PHYSICAL GEOGRAPHY. A Description of the most Remarkable Natural Phenomena. By WILLIAM WITTICH. New edition. Small 8vo, 2s. 6d. cloth.

WOLFE.—MESSIANIC PROPHECIES.

THE MESSIAH, as PREDICTED in the PENTATEUCH and PSALMS; being a new Translation and Critical Exposition of these Sacred Oracles. By J. R. WOLFE, author of the "Practical Hebrew Grammar." 8vo. 10s. 6d. cloth.

ZOOLOGY—ILLUSTRATIONS OF.

A Series of Ninety Engravings, comprehending about a Thousand Figures of Quadrupeds, Birds, Fish, Reptiles, Mollusca, Insects, Crustacea, Polyps, &c., engraved by J. W. Lowry and Thomas Landseer, after Sowerby, Charles Landseer, and others. The Descriptions by JOHN FLINT SOUTH, Esq., F.L.S., J. E. GRAY, Esq., F.L.S., J. F. STEPHENS, Esq., F.L.S., F.Z.S., and others. Im. 4to, 31s. 6d. cl. gt.

Cabinet Edition of the Encyclopædia Metropolitana—continued.

Vol. XXV., price 10s. 6d., cloth lettered,

HISTORY OF THE ROMAN EMPIRE. By the Rev. THOS. ARNOLD, D.D., late Head Master, Rugby School, and Others. With Numerous Illustrations.

Vol. XXVI., price 10s. 6d., cloth lettered,

HISTORY OF THE DECLINE AND FALL OF THE ROMAN POWER. By the Right Rev. Bishop RUSSEL, &c. With Numerous Illustrations.

Vol. XXVII., price 6s., cloth lettered,

HISTORY OF GREEK AND ROMAN PHILOSOPHY AND SCIENCE. By the Right Rev. CHARLES BLOMFIELD, D.D., Bishop of London; Rev. Dr. WHEWELL, and other Contributors.

"A more interesting and instructive volume than this has not been offered to the public for a long time."—*Critic.*

Vol. XXVIII., price 3s. 6d., cloth lettered,

MORAL AND METAPHYSICAL PHILOSOPHY—THE PHILOSOPHY OF THE FIRST SIX CENTURIES. By the Rev. F. D. MAURICE, M.A.

Vol. XXIX., price 7s. 6d., cloth lettered,

HISTORY OF THE OTTOMAN EMPIRE; including a Survey of the GREEK EMPIRE and the CRUSADES. By Colonel PROCTER, Rev. J. E. RIDDLE, and JAMES M'CONECHY, Esq.

Vol. XXX., price , cloth lettered,

TRIGONOMETRY. By GEORGE B. AIRY, F.R.S., Astronomer Royal. Edited by HUGH BLACKBURN, M.A., Professor of Mathematics in the University of Glasgow.

Vol. XXXI., price , cloth lettered,

HISTORY OF THE OCCULT SCIENCES. By the Rev. EDWARD SMEDLEY, M.A., E. RICH, Esq., and other Contributors. *(In March.)*

Vol. XXXII., price 12s. 6d., cloth lettered,

GEOLOGY—PRACTICAL AND THEORETICAL. By JOHN PHILLIPS, M.A., F.R.S., F.G.S., Deputy Reader in Geology in the University of Oxford. With Numerous Illustrations. *(In May.)*

Volumes in Active Preparation.

I.

SCULPTURE. By RICHARD WESTMACOTT, F.R.S., A.R.A., with Numerous Illustrations, revised.

II.

THE STEAM ENGINE: its History and Applications to Locomotion, Navigation, Mining, and Manufactures. By WILLIAM FERGUSON, F.C.S. With Numerous Illustrations.

III.

ECCLESIASTICAL HISTORY, from the Fourth to the Twelfth Century. By Rev. J. M. GUILDING.

IV.

ECCLESIASTICAL HISTORY, from the Thirteenth Century to the Present Day. By Rev. Dr. HAMPDEN, Bishop of Hereford; Rev. J. E. RIDDLE, and Others.

V.

GLOSSOLOGY: or, The Historical Relations of Languages. By SIR JOHN STODDART, LL.D.

VI.

MORAL AND METAPHYSICAL PHILOSOPHY—THE MIDDLE AGES. By Rev. F. D. MAURICE, M.A., Chaplain of Lincoln's Inn.

VII.

MORAL AND METAPHYSICAL PHILOSOPHY—MODERN PHILOSOPHY. By Rev. F. D. MAURICE, M.A., Chaplain of Lincoln's Inn.

VIII.

PRINCIPLES AND PRACTICE OF MEDICINE. By R. WILLIAMS, M.D., Physician to St. Thomas's Hospital, London, and W. AITKEN, M.D., Demonstrator of Anatomy in the University of Glasgow.

IX.

ARITHMETIC. By Rev DR. PEACOCK, Dean of Ely. Revised

GRIFFIN'S PORTABLE CYCLOPÆDIAS,

A Series of Reference Books,

ADAPTED TO THE WANTS OF THE PRESENT AGE,

EMBRACING BIOGRAPHY, GEOGRAPHY, HISTORY, RELIGION, THE NATURAL AND PHYSICAL SCIENCES, &c.

Handsomely printed in 8vo, and profusely Illustrated.

BIBLICAL CYCLOPÆDIA; or, Complete Dictionary of Eastern Antiquities, Geography, Natural History, Sacred Annals and Biography, Theology, and Biblical Literature. By the Rev. JOHN EADIE, D.D., LL.D., Professor of Biblical Literature to the United Presbyterian Church. With Numerous Illustrations. Fifth Edition, revised, 7s. 6d.

*** Whatever could be regarded as sectarian by any denomination of evangelical Christians (whether Episcopalian, Presbyterian, or Congregational) is scrupulously excluded, especially in all that relates to ritual and government. The volume is thus in accordance with the spirit and principles of the "Evangelical Alliance."

CYCLOPÆDIA OF CHEMISTRY: including the Application of the Science to the ARTS, MINERALOGY, and PHYSIOLOGY. By ROBERT DUNDAS THOMSON, M.D., F.R.S., F.R.S.E., F.C.S., &c., Professor of Chemistry at St. Thomas's Hospital College, London. With Numerous Illustrations, 12s. 6d.

"Its limited compass—its completeness and correctness, recommend the Cyclopædia of Chemistry to all who feel an interest in the science."—*Athenæum.*

"In every respect worthy of praise. It is within a limited space by far the best compendium of Chemistry which we have yet seen."—*Lancet.*

"A practical publication in the highest degree valuable."—*Mining Journal.*

"Contains a comprehensive summary of Chemical science and arts in a form most useful for reference and suitable for study."—*Literary Gazette.*

CYCLOPÆDIA OF UNIVERSAL BIOGRAPHY; containing a Series of Original Memoirs of the most remarkable Individuals of all Times and Nations. Edited by E. RICH, Esq., with the assistance of numerous Contributors. With 150 Illustrations of Birth-Places, &c. &c. 12s. 6d.

NAMES OF THE CONTRIBUTORS.

History.

SIR ARCHIBALD ALISON, D.C.L.
JOHN HILL BURTON, ESQ.
PROFESSOR CREASY, M.A.
PROFESSOR FERGUSON, A.M.
E. RICH, ESQ.

Theology and Literature.

THOMAS DAVIDSON, ESQ.
PROFESSOR EADIE, D.D., LL.D.
JAMES HEDDERWICK, ESQ.
ROBERT JAMIESON, D.D.
CHARLES KNIGHT, ESQ.
PROFESSOR SPALDING, M.A.

Natural and Physical Sciences.

W. BAIRD, M.D., F.L.S.
SIR DAVID BREWSTER, F.R.S.
JAMES BRYCE, ESQ., F.G.S.
PROFESSOR GORDON, F.R.S.
PROFESSOR NICHOL, LL.D.
R. D. THOMSON, M.D., F.R.S.
JAMES M'CONECHY.

Fine Arts.

JOHN A. HERAUD, ESQ.
JAMES MANSON, ESQ.
RALPH N. WORNUM, ESQ.

"A wonderful instance of cheapness, completeness, and excellence."—*Critic.*

"This work differs from all single volumes of Biography in several remarkable features;—the great number of subjects which it contains, the variety and indeed celebrity of the writers engaged, and the original character imparted to the larger notices. The first, and for purposes of reference the most useful feature, is the number of persons noticed: this is managed by very brief accounts of lesser names, in great numbers of cases by little more than lists with the date of birth and death, and the circumstance, action, or work, which procures the subject a place. The men engaged on the biographies are numerous and celebrated, as may be seen from the enumeration in the title; and each writer takes that class of lives with which his studies have made him most familiar. Alison and Creasy, for example, undertake naval and military men. This distribution gives a striking and novel peculiarity to the lives of prominent men, however short. It gets rid of mere dry compilation. The writers take a whole view of the life and character of their subjects, and present a judgment on both, instead of a bare recital of facts with an echo of opinions so old as to be commonplace. The scale is well preserved."—*Spectator.*

"No English work of the kind can stand a comparison with the present one."—*Manchester Advertiser.*

"With those articles to which we have turned in order to test the book we are highly satisfied."—*Literary Gazette.*

"The *ne plus ultra* of biographical dictionaries."—*Glasgow Herald.*

CYCLOPÆDIA OF HISTORY: a Complete Series of Chronological Tables, from the Beginning of the World to the Present Time, on a New Plan, comprehending an arrangement according to dates and an Alphabetical arrangement. 10s. 6d.

*** This Work will be found to be the "*very marrow of History*," and to contain a vast amount of information interesting to every one (and who is not?) interested in the progress of the human race. From the peculiarity of its arrangement, the events which have taken place at any particular time may be seen [illegible] most extensive Index, speedy information can be obtaine[illegible]

2s 6d